MATH CIRCLE CONTEST II

December 12, 2001

1. HOLIDAY HEADACHES

The holiday choir director is trying to arrange his seven member choir (Andrew, Ben, Cody, David, Elizabeth, Fran, Gina) into a row. Because of the vocal capabilities of each member and the acoustics of the room, the director decides that *at least one of the following conditions must hold*

(a) Andrew must be to the left of Ben

(b) Cody must be to the left of Fran

(c) Fran must be to the left of Gina.

Given this stipulation, how many possible arrangements can the director make?

2. VARIATIONS ON A THEME OF CATALAN

Recall that the nth Catalan number C_n is defined to be the number of ways from moving on a grid from $(0,0)$ to (n,n) by moves one unit up or to the right, none of which cross the diagonal. We set $C_0 = 1$.

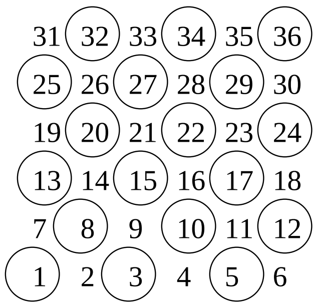
(a) Suppose that we instead are interested in paths that cross the diagonal *exactly twice*, but which are never more than one move above the diagonal. Prove that the number of such paths from $(0,0)$ to (n,n) is

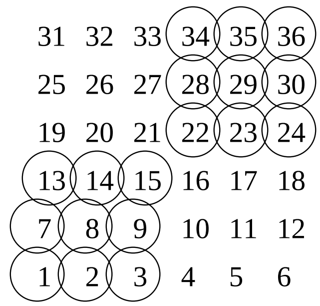
$$\sum_{j=0}^{n-1} C_j C_{n-1-j}.$$

(b) Now suppose that we are instead interested in paths that cross the diagonal *exactly once*. Prove that the number of such paths from $(0,0)$ to (n,n) is

$$\sum_{j=1}^{n-1} C_j C_{n-j}.$$

Consider the following two pictures.





The sum of the circled boxes in the left hand figure equals the sum of the uncircled boxes. (This common value is 333.) This is the same as the sum of the circled numbers in the right figure, as well as the sum of the uncircled boxes in the right figure.

Now consider two 20-by-20 squares with entries $1, \ldots, 400$ arranged as above. In each square circle half of the numbers according to the above pictures. Prove that the sum of the circled numbers in one square is the same as the number of the circled numbers in the other. What is this number?