Introduction to Graph Theory

Allen Dickson

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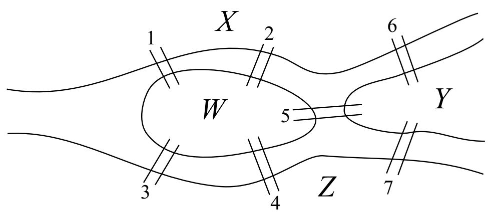
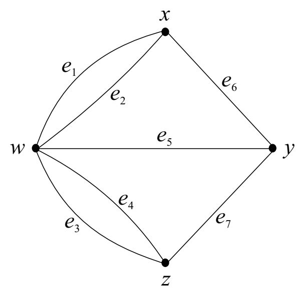
1 The Königsberg Bridge Problem

The city of Königsberg was located on the Pregel river in Prussia. The river divided the city into four separate landmasses, including the island of Kneiphopf. These four regions were linked by seven bridges as shown in the diagram. Residents of the city wondered if it were possible to leave home, cross each of the seven bridges exactly once, and return home. The Swiss mathematician Leonhard Euler (1707-1783) thought about this problem and the method he used to solve it is considered by many to be the birth of graph theory.





Exercise 1.1. See if you can find a round trip through the city crossing each bridge exactly once, or try to explain why such a trip is not possible.

The key to Euler's solution was in a very simple abstraction of the puzzle. Let us redraw our diagram of the city of Königsberg by representing each of the land masses as a vertex and representing each bridge as an edge connecting the vertices corresponding to the land masses. We now have a graph that encodes the necessary information. The problem reduces to finding a "closed walk" in the graph which traverses each edge exactly once, this is called an Eulerian circuit. Does such a circuit exist?

2 Fundamental Definitions

We will make the ideas of graphs and circuits from the Königsberg Bridge problem more precise by providing rigorous mathematical definitions.

A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge, two vertices called its **endpoints** (not necessarily distinct).

Graphically, we represent a graph by drawing a point for each vertex and representing each edge by a curve joining its endpoints.

For our purposes all graphs will be **finite graphs**, i.e. graphs for which $V(G)$ and $E(G)$ are finite sets, unless specifically stated otherwise.

Note that in our definition, we do not exclude the possibility that the two endpoints of an edge are the same vertex. This is called a **loop**, for obvious reasons. Also, we may have **multiple edges**, which is when more than one edge shares the same set of endpoints, i.e. the edges of the graph are not uniquely determined by their endpoints.

A **simple graph** is a graph having no loops or multiple edges. In this case, each edge e in $E(G)$ can be specified by its endpoints u, v in $V(G)$. Sometimes we write $e = uv$.

When two vertices u, v in $V(G)$ are endpoints of an edge, we say u and v are **adjacent**.

A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the ordering. A path which begins at vertex u and ends at vertex v is called a u, v-path.

A **cycle** is a simple graph whose vertices can be cyclically ordered so that two vertices are adjacent if and only if they are consecutive in the cyclic ordering.

We usually think of paths and cycles as subgraphs within some larger graph.

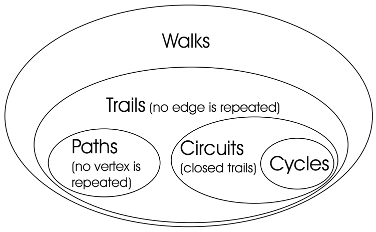
A **subgraph** H of a graph G, is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ satisfying the property that for every $e \in E(H)$, where e has endpoints $u, v \in V(G)$ in the graph G, then $u, v \in V(H)$ and e has endpoints u, v in H, i.e. the edge relation in H is the same as in G.

A graph G is **connected** if for every $u, v \in V(G)$ there exists a u, v-path in G. Otherwise G is called **disconnected**. The maximal connected subgraphs of G are called its **components**.

A **walk** is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A **trail** is a walk with no repeated edge. A u, v-walk or u, v-trail has first vertex u and last vertex v. When the first and last vertex of a walk or trail are the same, we say that they are closed. A closed trail is called a **circuit**.

With this new terminology, we can consider paths and cycles not just as subgraphs, but also as ordered lists of vertices and edges. From this point of view, a path is a trail with no repeated vertex, and a cycle is a closed trail (circuit) with no repeated vertex other than the first vertex equals the last vertex.

Alternatively, we could consider the subgraph traced out by a walk or trail.

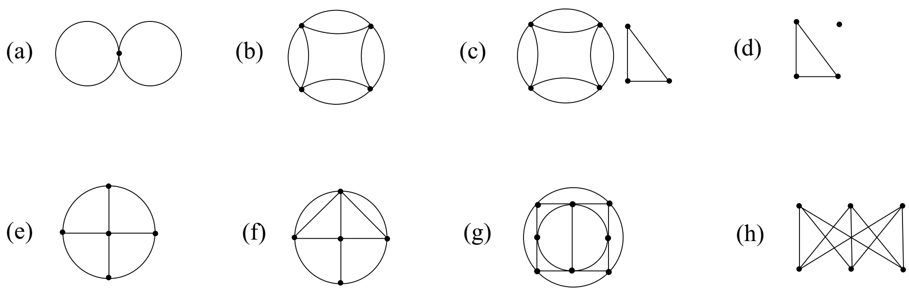


An **Eulerian trail** is a trail in the graph which contains all of the edges of the graph. An **Eulerian circuit** is a circuit in the graph which contains all of the edges of the graph. A graph is **Eulerian** if it has an Eulerian circuit.

The **degree** of a vertex v in a graph G, denoted $\deg v$, is the number of edges in G which have v as an endpoint.

3 Exercises

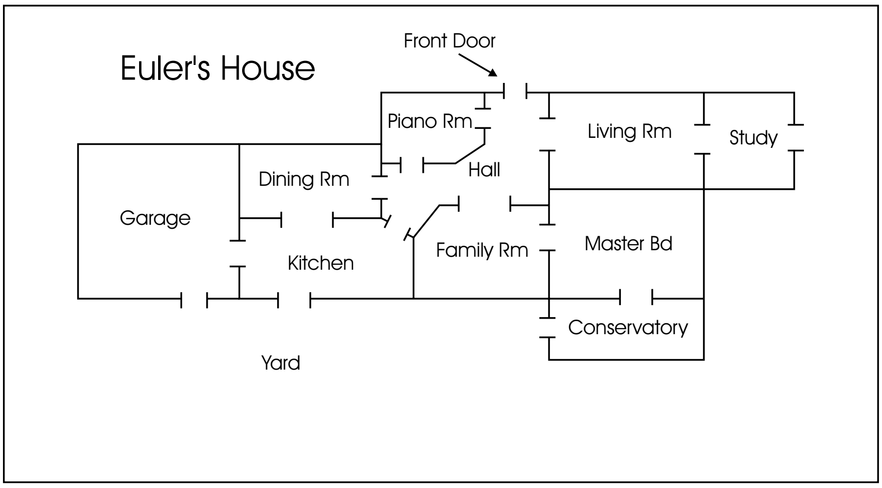
Consider the following collection of graphs:



1. Which graphs are simple?
2. Suppose that for any graph, we decide to add a loop to one of the vertices. Does this affect whether or not the graph is Eulerian?
3. Which graphs are connected?
4. Which graphs are Eulerian? Trace out an Eulerian circuit or explain why an Eulerian circuit is not possible.
5. Are there any graphs above that are not Eulerian, but have an Eulerian trail?
6. Give necessary conditions for a graph to be Eulerian.
7. Give necessary conditions for a graph to have an Eulerian trail.

8. Given that a graph has an Eulerian circuit beginning and ending at a vertex v, is it possible to construct an Eulerian circuit beginning and ending at any vertex in the graph?

9. Euler's House. Baby Euler has just learned to walk. He is curious to know if he can walk through every doorway in his house exactly once, and return to the room he started in. Will baby Euler succeed? Can baby Euler walk through every door exactly once and return to a different place than where he started? What if the front door is closed?



4 Characterization of Eulerian Circuits

We have seen that there are two obvious necessary conditions for a graph to be Eulerian: the graph must have at most one nontrivial component, and every vertex in the graph must have even degree. Now a more interesting question is, are these conditions sufficient? That is, does every connected graph with vertices of even degree have an Eulerian circuit? This is the more difficult question which Euler was able to prove in the affirmative.

Theorem 1. *A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.*

There are at least three different approaches to the proof of this theorem. We will use a constructive proof that provides the most insight to the problem. There is also a nonconstructive proof using maximality, and a proof that implements an algorithm.

We will need the following result.

Lemma 2. *If every vertex of a graph G has degree at least 2, then G contains a cycle.*

Proof. Let P be a maximal path in G. Maximal means that the path P cannot be extended to form a larger path. Why does such a path exist? Now let u be an endpoint of P. Since P is maximal (cannot be extended), every vertex adjacent to u must already be in P. Since u has degree at least two, there is an edge e extending from u to some other vertex v in P, where e is not in P. The edge e together with the section of P from u to v completes a cycle in G. ■

Proof of theorem. We have already seen that if G is Eulerian, then G has at most one nontrivial component and all of the vertices of G have even degree. We just need to prove the converse.

Suppose G has at most one nontrivial component and that all of the vertices of G have even degree. We will use induction on the number of edges n.

Basis step: When $n = 0$, a circuit consisting of just one vertex is an Eulerian circuit.

Induction step: When $n > 0$, each vertex in the nontrivial component of G has degree at least 2. Why? By the lemma, there is a cycle C in the nontrivial component of G. Let G' be the graph obtained from G by deleting $E(C)$. Note that G' is a subgraph of G which also has the property that all of its vertices have even degree. Why? Note also that G' may have several nontrivial components. Each of these components of G' must have number of edges $< n$. Why? By the induction hypothesis, each of these components has an Eulerian circuit. To construct an Eulerian circuit for G, we traverse the cycle C, but when a component of G' is entered for the first time (why must every component intersect C?), we detour along an Eulerian circuit of that component. This circuit ends at the vertex where we began the detour. When we complete the traversal of C, we have completed the Eulerian circuit of G. ■

5 The Degree-Sum Formula and the Handshaking Lemma

Proposition 3. *(Degree-Sum Formula) If G is a graph, then*

$$\sum_{v \in V(G)} \deg v = 2 \cdot \#E(G)$$

where $\#E(G)$ is the number of edges in G.

Proof. This is simply a matter of counting each edge twice. The details are left as an exercise. ■

This formula is extremely useful in many applications where the number of vertices and number of edges are involved in calculations. For example, we will learn later about the graph invariants of Euler characteristic and genus; the degree-sum formula often allows us to prove inequalities bounding the values of these invariants.

A fun corollary of the degree-sum formula is the following statement, also known as the handshaking lemma.

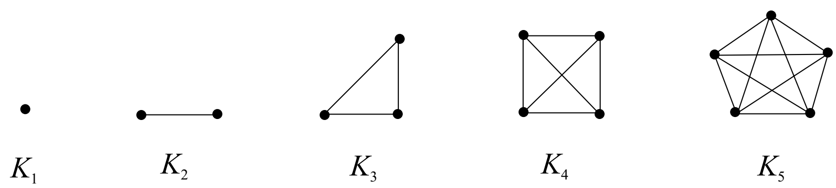
Corollary 4. *In any graph, the number of vertices of odd degree is even. Or equivalently, the number of people in the universe who have shaken hands with an odd number of people is even.*

Proof. Try to solve this one yourself.

Hint: Split the sum on the left hand side of the degree-sum formula into two pieces–one over vertices of even degree and one over vertices of odd degree. ■

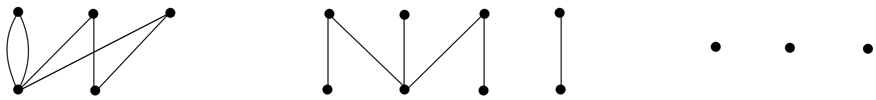
6 Important Graphs

There are two special types of graphs which play a central role in graph theory, they are the complete graphs and the complete bipartite graphs. A **complete graph** is a simple graph whose vertices are pairwise adjacent. The complete graph with n vertices is denoted K_n.

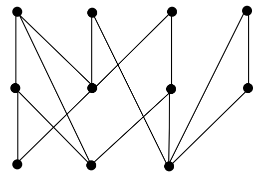


Before we can talk about complete bipartite graphs, we must understand bipartite graphs. An **independent set** in a graph is a set of vertices that are pairwise nonadjacent. A graph G is **bipartite** if $V(G)$ is the union of two disjoint (possibly empty) independent sets, called **partite sets** of G.

Similarly, a graph is k-partite if $V(G)$ is the union of k disjoint independent sets.

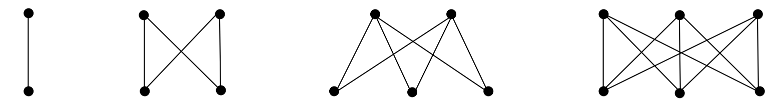


3 different bipartite graphs



A 3-partite graph

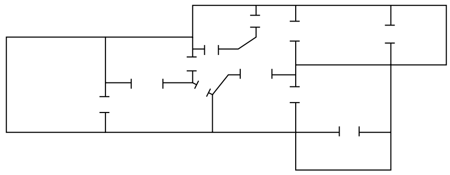
A **complete bipartite graph** is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. The complete bipartite graph with partite sets of size m and n is denoted $K_{m,n}$.



$K_{1,1}$ $K_{2,2}$ $K_{2,3}$ $K_{3,3}$

7 Exercises

1. Determine the values of m and n such that $K_{m,n}$ is Eulerian
2. Prove or disprove:
   (a) Every Eulerian bipartite graph has an even number of edges.
   (b) Every Eulerian simple graph with an even number of vertices has an even number of edges. What if we also assume that the graph has only one component?
3. When is a cycle a bipartite graph?
4. Oh no! Baby Euler has gotten into the handpaints. His favorite colors are blue and yellow. Baby Euler wants to paint each room in the house (including the hall) either blue or yellow such that every time he walks from one room to an adjacent room, the color changes. Is this possible?



5. If we consider the graph corresponding to Euler's house, the previous problem is equivalent to assigning the color blue or yellow to each vertex of the graph so that no two vertices of the same color are adjacent. This is called a 2-coloring of the graph. What is the relationship between 2-coloring vertices of a graph and bipartite graphs?

8 k-partite and k-colorable

A k-**coloring** of a graph G, is a labeling of the vertices $f : V(G) \to S$, where S is some set such that $|S| = k$. Normally we think of the set S as a collection of k different colors, say $S = \{\text{red, blue, green, etc.}\}$, or more abstractly as the positive integers $S = \{1, 2, \ldots, k\}$. The labels are called **colors**. A k-coloring is **proper** if adjacent vertices are different colors. A graph is k-**colorable** if it has a proper k-coloring. The **chromatic number** $\chi(G)$ is the least positive integer k such that G is k-colorable.

You should notice that a graph is k-colorable if and only if it is k-partite. In other words, k-colorable and k-partite mean the same thing. You should convince yourself of this by determining the k different partite sets of a k-colorable graph and conversely determine a k-coloring of a k-partite graph.

In general it is not easy to determine the chromatic number of a graph or even if a graph is k-colorable for a given k.

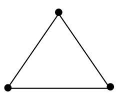
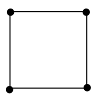
9 Exercises

1. If a graph G is k-partite, what do we know about $\chi(G)$?

2. Show that $\chi(G) = 1$ if and only if G is totally disconnected, i.e. all of the components of G contain only 1 vertex.

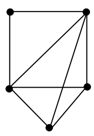
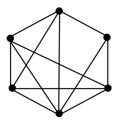
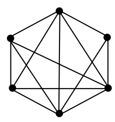
3. For a finite graph G, is $\chi(G)$ also finite? Find an upper bound, or find a finite graph G which cannot be colored by finitely many colors.

4. Determine the chromatic number for each of the following graphs:

(a)  (b) 

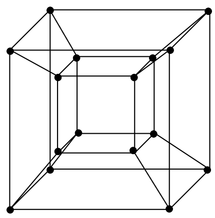
(c) Any cycle of odd length (the **length** of a cycle is the number of edges in the cycle).

(d) Any cycle of even length.

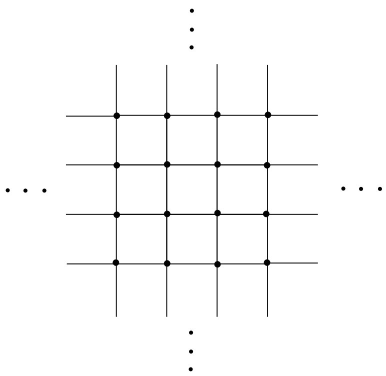
(e)  (f)  (g) 

(h) The graph $K_{m,n}$

(i) The graph K_n

(j) 

This graph is called the hypercube, or 4-dimensional cube.



(k)

This is an example of an infinite graph. If the vertices of the graph are the integer coordinates, then this is also an example of a unit distance graph, since two vertices are adjacent if and only if they are distance one apart.

5. Suppose that the graph G is bipartite, i.e. 2-colorable, is it possible for G to contain a cycle of odd length?

10 Characterization of Bipartite Graphs

We have just seen that if G is bipartite, then G contains no cycles of odd length. Equivalently, this means that if G does have a cycle of odd length, then G is not bipartite, hence not 2-colorable (this is the contra-positive of the previous statement). You should look back at the problem of coloring the rooms in Euler's house to determine if there is such an odd cycle.

The obvious question now is whether or not the converse of the above statement is true. That is, if G contains no cycles of odd length, does it hold that G is bipartite? The answer is yes.

Theorem 5. *A graph G is bipartite if and only if G contains no cycles of odd length.*

Proof. You should have already proved the forward direction in the exercises, so we will prove the other direction. Suppose that G contains no cycles of odd length. We might as well assume that G contains only one component, since if it has more than one, we can form a bipartition of the graph from the bipartition of each of its components. Thus assume G has one component. Pick a vertex $u \in V(G)$. For each $v \in V(G)$, let $f(v)$ be the minimum length of a u, v-path. Since G is connected, $f(v)$ is defined for each $v \in V(G)$. Let $X = \{v \in V(G) \mid f(v) \text{ is even}\}$ and $Y = \{v \in V(G) \mid f(v) \text{ is odd}\}$. We wish to show that X and Y are independent sets of vertices. Indeed, if there are adjacent vertices v, v' both in X (or both in Y), then the closed walk consisting of the shortest u, v-path, plus the edge v to v', plus the reverse of the shortest u, v'-path, is a closed walk of odd length. It can be shown by induction that every closed walk of odd length contains an odd cycle, but this contradicts our hypothesis that G contains no cycles of odd length. Therefore no two vertices in X (or Y) are adjacent, i.e. X and Y are independent so that G is bipartite. ■

Exercise 10.1. Show that every closed walk of odd length contains a cycle of odd length.

Hint: Use induction on the length l of the closed walk. If the closed walk has no repeated vertex, then it is a cycle of odd length. If it does have a repeated vertex v, then break the closed walk into two shorter walks.

11 Upper and Lower Bounds for $\chi(G)$

We have already seen an upper bound for $\chi(G)$ in the exercises, namely

$$\chi(G) \leq \#V(G).$$

For the particular case of the complete graph we have $\chi(K_n) = \#V(K_n) = n$, so this is the best possible upper bound for the chromatic number in terms of the size of the vertex set. However, we may derive other upper bounds using other structural information about the graph. As an example, we will show that

$$\chi(G) \leq \triangle(G) + 1,$$

where the number $\triangle(G)$ is the maximum degree of all the vertices of G.

Begin with an ordering $v_1, v_2, \ldots, v_n$ of all of the vertices of G. The **greedy coloring** of G colors the vertices in order $v_1, v_2, \ldots v_n$ and assigns to each v_i the lowest-indexed color which has not already been assigned to any of the previous vertices in the ordering that are adjacent to v_i. Note that in any vertex ordering of G, each vertex v_i has at most $\triangle(G)$ vertices which are adjacent to v_i and have appeared earlier in the ordering. Thus as we color each vertex of G, we never need more than $\triangle(G)+1$ colors. It follows that

$$\chi(G) \leq \triangle(G) + 1.$$

To give a useful lower bound, we define a set of vertices called a clique, which is complementary to the notion of an independent set defined earlier. A **clique** in a graph is a set of pairwise adjacent vertices. The **clique number** of a graph G, denoted $\omega(G)$, is the maximum size of a clique in G.

In effect, a clique corresponds to a subgraph (whose vertices are the vertices of the clique) that is itself a complete graph. Thus if $\omega(G) = n$, then there is a clique of size n corresponding to a subgraph of G that is equivalent to K_n. Since it will require at least n colors to color the vertices in this clique, we have that

$$\chi(G) \geq \omega(G).$$

12 Unit Distance Graphs–An Open Problem

A **unit distance graph** is quite simply a graph whose vertices are points in the plane (or more generally any Euclidean space), with an edge between two vertices u and v if and only if the distance between u and v is 1.

Consider the unit distance graph whose set of vertices is the entire plane. Let us denote this graph by P. This is definitely not a finite graph, as there are uncountably many vertices, and for each vertex v there are uncountably many edges having v as an endpoint! In particular, for a given vertex v which corresponds to a point $(x, y) \in \mathbb{R}^2$ in the plane, the vertices which are adjacent to v are those which correspond to the points lying on the circle of radius 1 centered at (x, y).

Exercise 12.1. What is $\chi(P)$?

This number is also known as the chromatic number of the plane. This question can be restated more simply as follows:

How many colors are needed so that if each point in the plane is assigned one of the colors, no two points which are exactly distance 1 apart will be assigned the same color?

This problem has been open since 1956 and it is known that the answer is either 4, 5, 6 or 7 (apparently it is not very difficult to prove that these are the only possibilities). I was able to show rather easily that the answer is 3, 4, 5, 6, or 7 but I did not spend enough time working on the problem to determine how difficult it would be to eliminate 3 as a possibility. I encourage you to work on this yourself to get a feel for the subtleties of the problem. I don't think you will need much help eliminating 1 or 2. To prove that $\chi(P) \leq 9$, try tiling the plane with $\frac{1}{2} \times \frac{1}{2}$ squares and coloring the squares in a clever pattern. To show $\chi(P) \leq 7$, use a similar technique of tiling the plane into colored shapes. To eliminate 3 you will probably need to expand on the type of argument used to eliminate 2, but you're on your own here. Eliminate any of the remaining numbers and you can publish your results. To learn more about this and related open problems in graph theory, visit http://dimacs.rutgers.edu/~hochberg/undopen/graphtheory/graphtheory.html .

13 The Four Color Theorem and Planar Graphs

Arguably the most famous theorem in the field of graph theory is the Four Color Theorem. For an excellent history and explanation of the problem, see the article in Wikipedia at http://en.wikipedia.org/wiki/Four_color_theorem.

Briefly, this theorem states that 4 colors are sufficient to color regions in the plane so that no two regions which border each other have the same color. It is trivial to verify that 3 colors is not sufficient, and the proof that 5 colors is sufficient is not difficult. That 4 colors is indeed sufficient to color any subdivision of the plane, proved to be an extremely difficult problem that was finally solved in 1976 with the aid of a computer. This computer-aided proof has proved to be quite unsatisfying to many mathematicians.

The four color theorem can be stated quite simply in terms of graph theory. Just as with the Königsberg bridge problem, or the exercise about Euler's house, we abstract by representing the important information with a graph. Each region in the plane is represented as a vertex; two vertices are adjacent if and only if their corresponding regions border each other; and coloring the regions corresponds to a proper coloring of the vertices of the graph. You should notice that all possible graphs formed from such planar regions share an important property, namely they can be drawn in the plane without having to cross edges. This motivates the definition of planar graphs.

A graph is **planar** if it can be drawn in the plane without crossings.

(Examples of planar and nonplanar graphs.)

Theorem 6. *(Four color theorem–originally stated by P.J. Heawood 1890) For any planar graph G, we have $\chi(G) \leq 4$.*

Proof. K. Appel and W. Haken 1976. ■

14 Exercises

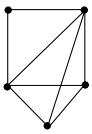
1. Determine if the following graphs are planar or nonplanar.

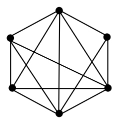
   (a) K_4

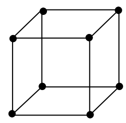
   (b) K_5

   (c) $K_{2,3}$

   (d) $K_{3,3}$

   (e) 

   (f) 

   (g) 

2. Find an example of a planar graph that is not 3-colorable.

3. Does the four color theorem imply that you need at most 4 colors to color a political map of the world so that each country is assigned a color and no two adjacent countries have the same color? Explain why or provide a counterexample. (This is a bit of a trick question)

4. Consider any planar graph G. Draw this graph in the plane so that there are no crossings. We refer to the regions of the plane bounded by the edges of the graph as **faces**, and denote the set of faces of G by $F(G)$. Compute
$$X(G) = \#V(G) - \#E(G) + \#F(G)$$
for each of the graphs (number of vertices minus the number of edges plus the number of faces). This is called the **Euler characteristic** of the graph. What trend do you notice?

5. (Kuratowski's theorem) Kuratowski proved that a finite graph is planar if and only if it contains no subgraph that is isomorphic to or is a subdivision of K_5 or $K_{3,3}$. In this sense, K_5 and $K_{3,3}$ are the basic building blocks of nonplanar graphs. Consider the graph from Exercise 1(f) above. Can you find a subgraph of this graph which looks like K_5 or $K_{3,3}$? What about the graph of the hypercube shown in Section 9?

15 The Genus of a Graph

The ability to draw a graph in the plane without crossings is equivalent to being able to draw a graph on a sphere without crossings. For example, if you can draw a graph on a sphere, simply puncture the sphere in the middle of one of the faces formed by the edges of the graph and then stretch out this hole until you can lay the sphere flat onto the plane. The result will be a drawing of the graph in the plane with no crossings. Conversely, if you can draw a graph in the plane without crossings, take the outer face (the face containing ∞) and reverse the process above by in essence wrapping the plane around the sphere (the point at ∞ corresponds to the punctured hole in the sphere).

The sphere is what we call a surface of genus zero. The genus of the surface tells you how many doughnut holes are in the surface. Thus a sphere has genus zero, a torus has genus one, a two-holed torus has genus 2, and so on. There are nonplanar graphs (hence cannot be drawn on a sphere without crossings), that can however be drawn on a torus without crossings. The graph K_5 is such a graph. Similarly, there are graphs which cannot be drawn on a torus but can be drawn on a two-holed torus. The minimum genus of surface upon which a graph can be drawn without crossing edges is called the **genus** of a graph and is denoted $\gamma(G)$. It can be shown that any finite graph can be drawn without crossings on a surface of large enough genus. Therefore the genus of a finite graph is well-defined.

The genus of most important graphs has been calculated. For example

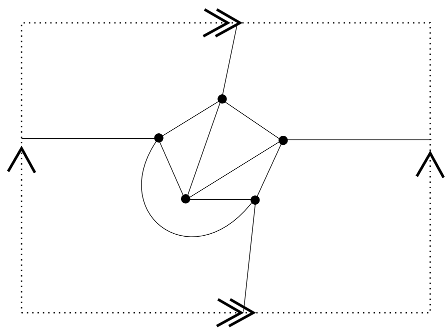
$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

and

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$$

where $\lceil \cdot \rceil$ denotes the ceiling function (these calculations can be found in Harary's book on graph theory). So for example, $\gamma(K_4) = 0, \gamma(K_5) = 1, \gamma(K_7) = 1$ and $\gamma(K_8) = 2$. Thus K_4 is the largest complete graph which can be drawn on the sphere and K_7 is the largest complete graph which can be drawn on the torus.

Exercise 15.1. The accompanying figure shows how to draw K_5 on the torus without crossing edges. Try to draw K_6 or K_7 on the torus.



K_5 on the torus