SOLUTIONS AND COMMENTS

The fields of algebra and number theory, besides being fundamental areas in mathematics, are huge fishing ponds for contest problems. The concepts of numbers, polynomials, functions are easy to understand, yet it's not hard to produce extremely complicated questions that require a high degree of cleverness and skill. It's hard to identify only a few universal key concepts or tricks. Modular arithmetic, factorization in primes, divisibility arguments, exponentials and logarithms, trig identities, polynomial simplification are all tools that you should have under your belt...and I could go on and on with more. In this session we have decided to concentrate on only a few concepts, to try and get an in depth comprehension and to be able to be handy and comfortable with them.

Modular arithmetic: rock around the clock

If you go to bed at 10pm and sleep for 9 hours (I wish...), at what time do you get up?
We are all able to answer: at 7 in the morning.

In doing so, we have performed an operation in modular arithmetic: we have added 9 to 10, but with the convention that, every time you reach 12, you "reset" and start counting from 0 again.

Mathematically speaking, this is called *counting modulo 12*. Of course there is nothing special about the number 12, and you can have fun counting modulo n for any number you want.

This can be an extremely handy tool to simplify a lot of computations during a contest. Let's see how we can use it to elegantly solve the first couple problems in the introduction.

Problem 1: Take any integer. Just to simplify notation let's suppose it's a 4 digit number *abcd*. Then, I can rewrite that same number as

$$a * 1000 + b * 100 + c * 10 + d.$$

Then let's observe that being divisible by 3 means exactly being 0 modulo three. So let us regard this same number modulo 3. Notice that all powers of 10 are equal to 1 modulo 3. Hence my number is exactly the same as

$$a + b + c + d.$$

And now we have accomplished the result. Do you see why?

Problem 2(a): The only possible real solutions to the equation are $x = 3, y = 2, z = -5$. It is sufficient to observe that an even power of a number is always nonnegative, and the sum of nonnegative numbers is zero only if all the summands are zero to begin with.

Problem 2(b): The polynomial does NOT have any integral root. The easy way to see it is again to dive into "modulo 3" world. Of course, if I have some integral solution, then I must have a modulo 3 solution as well. (notice that the converse is false: you can have solutions in modular arithmetic that do not correspond to real solutions: can you think of an example?). So now if we show that there aren't any solutions mod 3, we are home!

Observe first of all that my polynomial simplifies to

$$x^4 + 1$$

because 3 and 6 are 0 modulo 3. And now it suffices to substitute 0, 1, -1 in my expression and notice that I never get 0 (modulo 3).

Exponentials and logarithms

Exponentials and logarithms are some extremely fundamental and important tools in mathematics.

Given a number a and an integer n, a^n just means multiply a by itself exactly n times. From this definition follow a lot of properties and algebraic rules to manipulate exponentials that you should all be very familiar with. For example to you remember how to

1. take the product of two exponentials with the same base? ($a^m * a^n =?$)
2. take the product of two exponentials with the same exponent? ($a^m * b^m =?$)
3. exponentiate an exponential? ($(a^m)^n$)

Logarithms are, in some sense, the inverse creatures of exponents: We define $log_a b = c$ if $a^c = b$. Again it is of crucial importance to be completely familiar with how to manipulate logarithms. Let me just poke your memory...if you don't remember some of these rules, try and deduce them just from the properties of exponentials!

1. $log_a b + log_a c =?$;
2. $log_a(1/b) =?$;
3. $log_a(b^c) =?$;
4. what's the relationship between $log_a b$ and $log_b a$?

Problem 3: (a) is obviously correct. To see that all other answers are wrong it suffices to substitute $x = 2$.

Problem 4: Just by using the definition of logarithms, we can actually see that N is precisely

$$N = 7^{5^{3^{2^{11}}}}.$$

Hence the only prime factor is 7. The answer is 1.

Problem 5: Let's first of all do a common denominator between the first two terms, to get

$$\frac{a^2 + b^2}{ab} - ab.$$

Then we notice that if we add and subtract $2ab$ to the numerator of the fraction we get:

$$\frac{a^2 + b^2 - 2ab + 2ab}{ab} - ab = \frac{(a-b)^2 + 2ab}{ab} - ab.$$

But now we can use our relation $a - b = ab$ to get

$$\frac{(ab)^2 + 2ab}{ab} - ab = 2.$$