Recursion Relations and Qualitative Behavior of Sequences

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1 Introduction

Sequences play an important role in applied mathematics. They are often used as modeling tools for certain kinds of populations. When sequences are used in this manner, the general behavior of the sequence becomes an important quality that we would like to characterize. My goal today is to show you how to make these behavior assessments. I will reintroduce recursion relations and cobwebbing. Once this is complete, we will focus our energies on using cobwebbing to reveal the general behavior of sequences. In particular, we will discover that the behavior of any sequence defined by a recursion relation depends intimately on the recursion relation and initial condition.

2 What is a Recursion Relation?

A **sequence** is a particular ordering of objects that is indexed by the natural numbers ($\mathbb{N}$) or the non-negative integers ($\mathbb{Z}^+ + \{0\}$). Sequences can be composed of real numbers, functions, or sets and are often expressed as strings of these objects. Sometimes sequences can be expressed more formally by constructing a recursion relation. To do so, we let n be any natural number and a_n be the n^{th} term of some sequence.

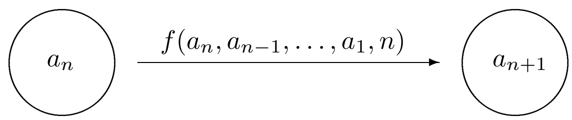
$$a_1, a_2, a_3, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots$$

A **recursion relation** defines the process one must go through to get from any term of the sequence, say a_n, to the next one, a_{n+1}. This process may depend on any or all of the previous sequence terms $(a_1, a_2, \ldots, a_n)$

as well as the index variable n. In the language of mathematics, recursion relations take on the form

$$a_{n+1} = f(a_n, a_{n-1}, \ldots, a_1, n)$$

where f is a function that defines the before-mentioned process. The function f is sometimes called an **updating function** as it uses known information about the sequence to produce new terms of the sequence.



Recursion relations alone cannot be used to represent a sequence. You must also provide additional information. An **initial condition** is any information that must be specified so that a recursion relation can be used to fully represent a sequence of interest.

Example 1. Find the first four terms of the sequence generated by $a_{n+1} = a_n^2$ beginning with $a_1 = \frac{3}{2}$. Note that the updating function here is $f(a_n) = a_n^2$ and the initial condition is $a_1 = \frac{3}{2}$.

Sequence Term	Sequence Value
a_n	$a_{n+1} = a_n^2, a_1 = \frac{3}{2}$
a_1	$\frac{3}{2}$
a_2	$(\frac{3}{2})^2 = \frac{9}{4}$
a_3	$(\frac{9}{4})^2 = \frac{81}{16}$
a_4	$(\frac{81}{16})^2 = \frac{6561}{256}$
$\vdots$	$\vdots$

Solution: $a_{n+1} = a_n^2$ with $a_1 = \frac{3}{2}$ represents the sequence $\frac{3}{2}, \frac{9}{4}, \frac{81}{16}, \frac{6561}{256}, \ldots$.

Example 2. Find the first four terms of the sequence generated by $a_{n+1} = a_n^2$ beginning with $a_1 = 1$. Note that the updating function here is $f(a_n) = a_n^2$ (again) and the initial condition is now $a_1 = 1$.

Sequence Term a_n	Sequence Value $a_{n+1} = a_n^2, a_1 = 1$
a_1	1
a_2	$(1)^2 = 1$
a_3	$(1)^2 = 1$
a_4	$(1)^2 = 1$
$\vdots$	$\vdots$

Solution: $a_{n+1} = a_n^2$ with $a_1 = 1$ represents the sequence $1, 1, 1, 1, \ldots$.

Aside: Note that in the above example we could have used other recursion relations to represent this sequence. For instance, $a_{n+1} = a_n$ beginning with $a_1 = 1$ would have done the trick. It will become clear to you shortly why I chose to use $a_{n+1} = a_n^2$ instead.

Example 3. Find the first four terms of the sequence generated by $a_{n+1} = a_n^2$ beginning with $a_1 = \frac{1}{2}$. Note that the updating function here is $f(a_n) = a_n^2$ (again) and the initial condition is now $a_1 = \frac{1}{2}$.

Sequence Term a_n	Sequence Value $a_{n+1} = a_n^2, a_1 = \frac{1}{2}$
a_1	$\frac{1}{2}$
a_2	$(\frac{1}{2})^2 = \frac{1}{4}$
a_3	$(\frac{1}{4})^2 = \frac{1}{16}$
a_4	$(\frac{1}{16})^2 = \frac{1}{256}$
$\vdots$	$\vdots$

Solution: $a_{n+1} = a_n^2$ with $a_1 = \frac{1}{2}$ represents the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$.

Let's look back at our work so far. We have shown that the recursion relation $a_{n+1} = a_n^2$ generates the sequences

$$\frac{3}{2}, \frac{9}{4}, \frac{81}{16}, \frac{6561}{256}, \ldots$$

$$1, 1, 1, 1, \ldots$$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$$

when we prescribe the initial conditions to be $a_1 = \frac{3}{2}$, $a_1 = 1$, and $a_1 = \frac{1}{2}$ respectively. Shown in figure 1 is graph of these sequences. There is something interesting to think about here. Each sequence

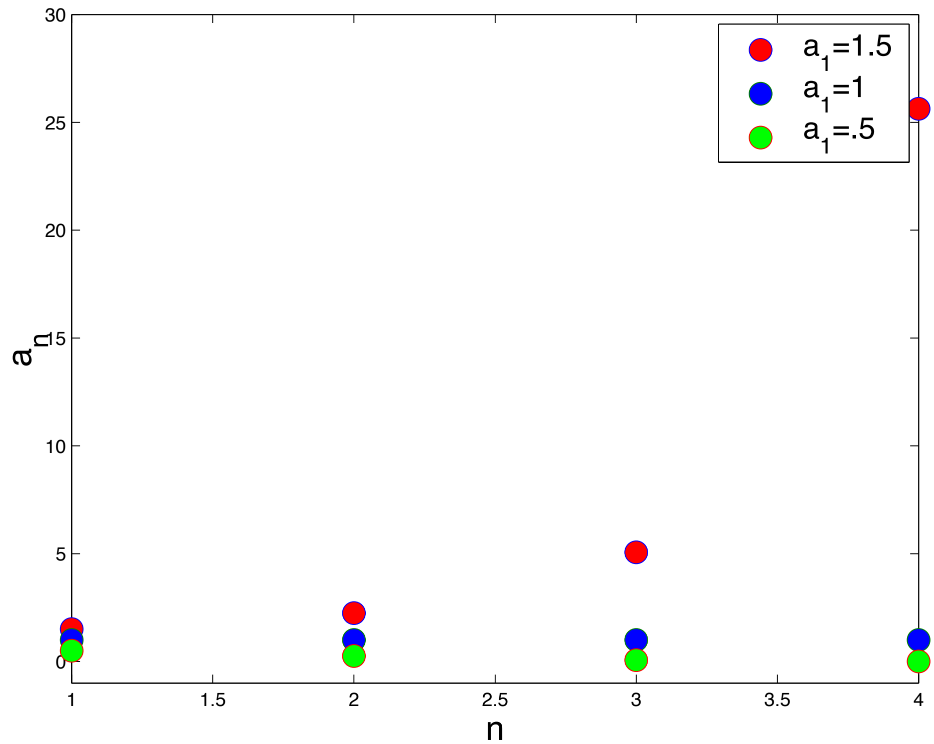


Figure 1: Graph of $a_{n+1} = a_n^2$ with the initial conditions $a_1 = \frac{3}{2}$, $a_1 = 1$, and $a_1 = \frac{1}{2}$

was generated from the same recursion relation, yet each has a very different feel to it. The sequence $\frac{3}{2}, \frac{9}{4}, \frac{81}{16}, \frac{6561}{256}, \ldots$ is increasing everywhere; the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$ is decreasing everywhere; the sequence $1, 1, 1, 1, \ldots$ remains constant always. This observation suggests that given some recursion relation $a_{n+1} = f(a_n, a_{n+1}, \ldots, a_1, n)$, the initial condition(s) we prescribe can have a resounding affect on the values the sequence takes on and the general trends the sequence exhibits.

Example 4. Find the first four terms of the sequence generated by $a_{n+1} = \frac{11}{10}a_n^2$ beginning with $a_1 = 1$. Note that the updating function here is $f(a_n) = \frac{11}{10}a_n^2$ and the initial condition is $a_1 = 1$.

Sequence Term a_n	Sequence Value $a_{n+1} = \frac{11}{10}a_n^2, a_1 = 1$
a_1	1
a_2	$\frac{11}{10}(1)^2 = \frac{11}{10}(1) = \frac{11}{10} = 1.1$
a_3	$\frac{11}{10}(\frac{11}{10})^2 = \frac{1331}{1000} = 1.331$
a_4	$\frac{11}{10}(\frac{1331}{1000})^2 = \frac{19487171}{10000000} = 1.9487171$
$\vdots$	$\vdots$

Solution: $a_{n+1} = a_n^2$ with $a_1 = 1$ represents the sequence $1, 1.1, 1.331, 1.9487171, \ldots$.

The purpose of Example 4 is to illustrate what can happen to sequence behavior when we alter the updating function, f, slightly. Notice that the sequences defined in Example 2 and Example 4 share the same initial condition ($a_1 = 1$) but have different updating functions ($f(a_n) = a_n^2$ and $f(a_n) = \frac{11}{10}a_n^2$). We found the resulting sequences to be

$$1, 1, 1, 1, \ldots$$

$$1, 1.1, 1.331, 1.9487171, \ldots$$

Again, we observe that these two sequences fail to share a general trend (even though their recursion relation representations are very similar). The sequence $1, 1, 1, 1, \ldots$ remains constant while the sequence $1, 1.1, 1.331, 1.9487171, \ldots$ appears to be increasing always. Consequently, we shouldn't be surprised if sequences generated by slightly different recursion relations behave differently.

The goal of this session is to continue thinking about the dependence of sequence behavior on initial conditions and updating functions. Instead of generating sequences explicitly using the recursion relation (coupled with an initial condition), we will learn a graphical method that reveals the general behavior of a sequence. This method focuses on revealing the relationship between consecutive terms without calculating these terms explicitly.

3 Cobwebbing Revisited

Cobwebbing is a graphical method that can be used to reveal the behavior of recursion relations taking on the form $a_{n+1} = f(a_n)$ when coupled with any initial condition a_1. The general idea is as follows:

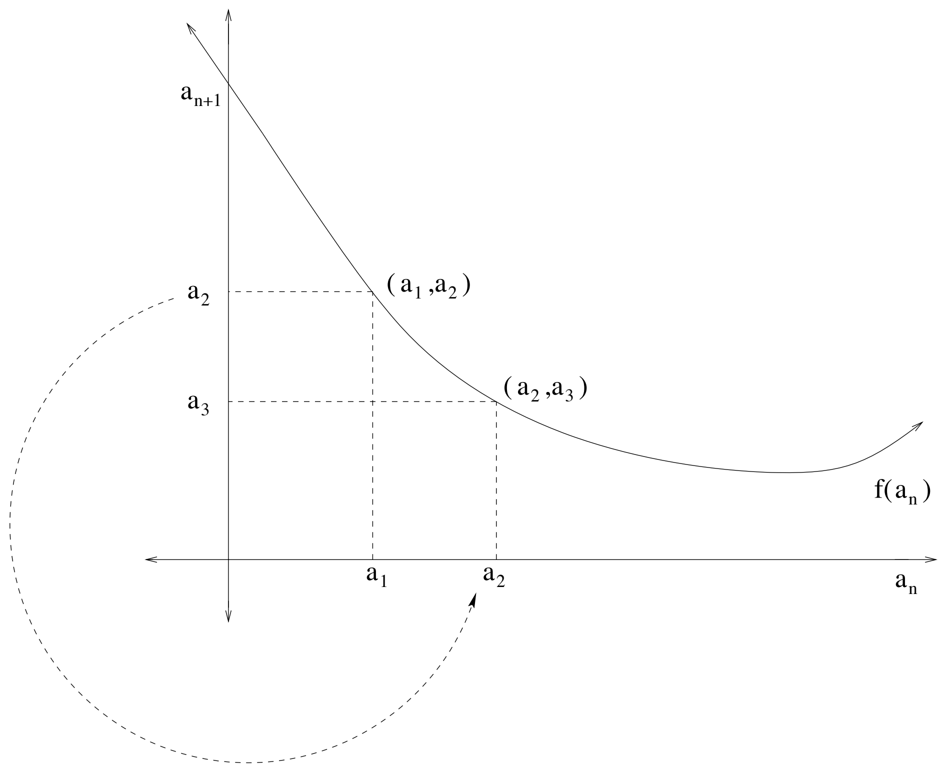


Figure 2: Graph of $a_{n+1} = f(a_n)$

1. To find a_2 from a_1, we use the fact that a_2 is the result of applying the updating function f to a_1. If we were to graph the updating function, we could visually identify a_2 as the y-coordinate associated with the point on the graph of the function directly above a_1 (see Figure 2).

2. Second, the axes of the above graph have special meaning. The horizontal axis represents the current term of the sequence while the vertical axis respresents the updated term. To get a_3 from a_2, we need to somehow get a_2 from the vertical axis to the horizontal axis while preserving its value. Once we have made this move, we can identify a_3 as y-coordinate associated with the point on the graph of the function directly above a_2 (see Figure 2).

3. What happens if we reflect terms from the vertical axis off the diagonal line $a_{n+1} = a_n$? Consider

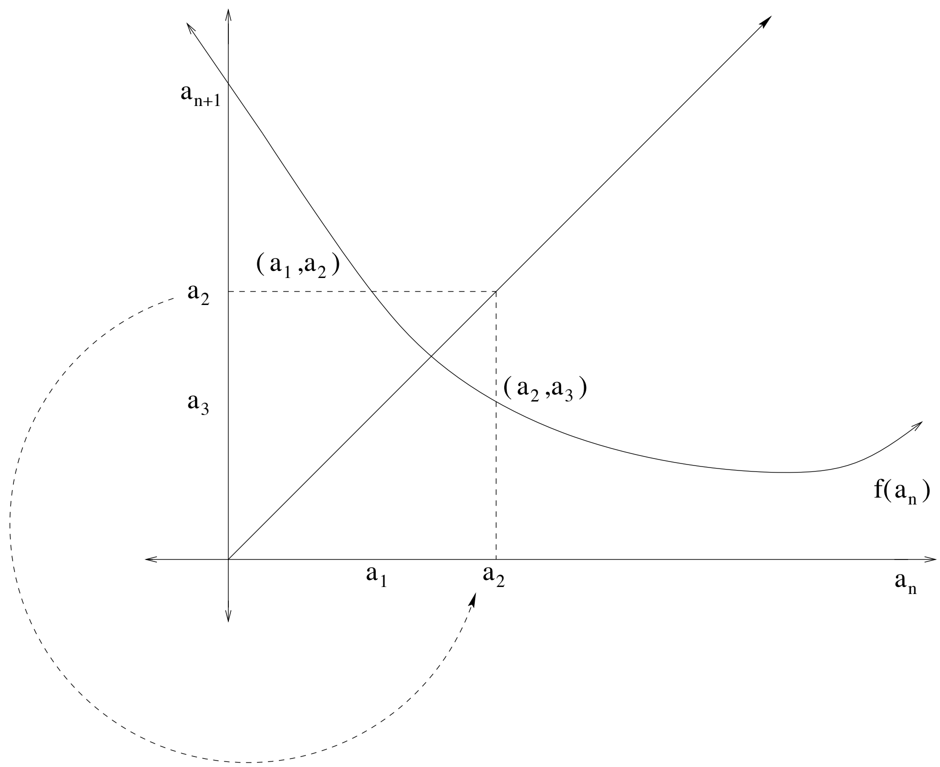


Figure 3: Graph of $a_{n+1} = f(a_n)$ and diagonal line $a_{n+1} = a_n$

reflecting a_2 from the vertical axis off this line. Notice in Figure 3 that this reflection places a_2 on the horizontal axis while preserving its value.

4. Now for the trick: move the point (a_1, a_2) horizontally until it intersects the diagonal line and then move vertically until you intersect the updating function (see Figure 4). These directions move the point (a_1, a_2) horizontally to (a_2, a_2) and vertically to (a_2, a_3). This trick takes advantage of our knowledge of a_2 to expose a_3. Cool, huh!?!

5. Repeat the previous step with a_3 to get a_4 (and in general with a_n to get a_{n+1}).

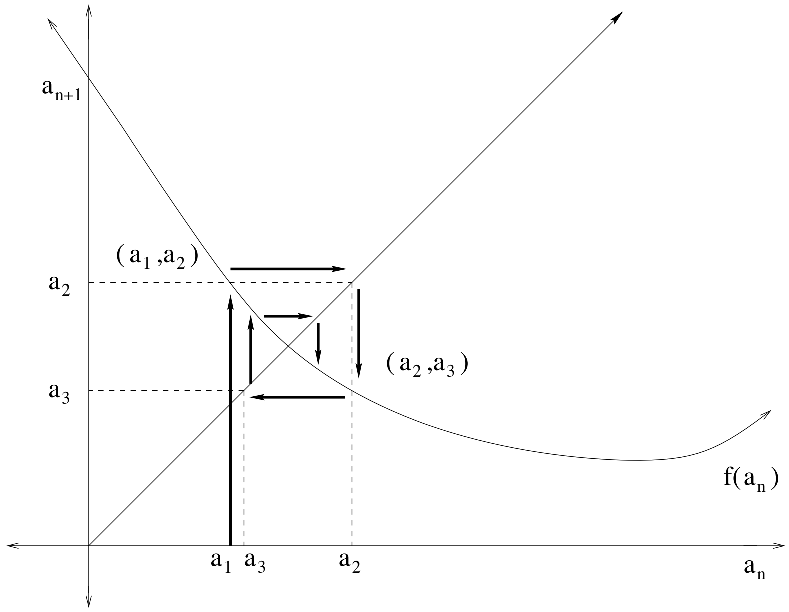


Figure 4: Cobwebbing Technique

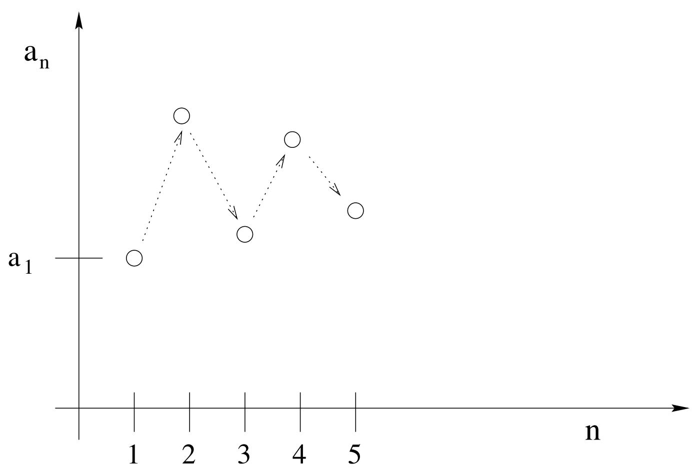


Figure 5: Sketch of a_n

If we are careful to mark the value of each sequence term on the horizontal axis as we go, we can use this information to provide a sketch of the sequence. Doing so will expose the general behavior of the sequence without explicitly calculating terms of the sequence (see Figure 5).

Example 5. Discuss the behavior of the sequences generated by $a_{n+1} = a_n^2$ when coupled with the initial conditions $a_1 = \frac{3}{2}$, $a_1 = 1$, and $a_1 = \frac{1}{2}$ respectively.

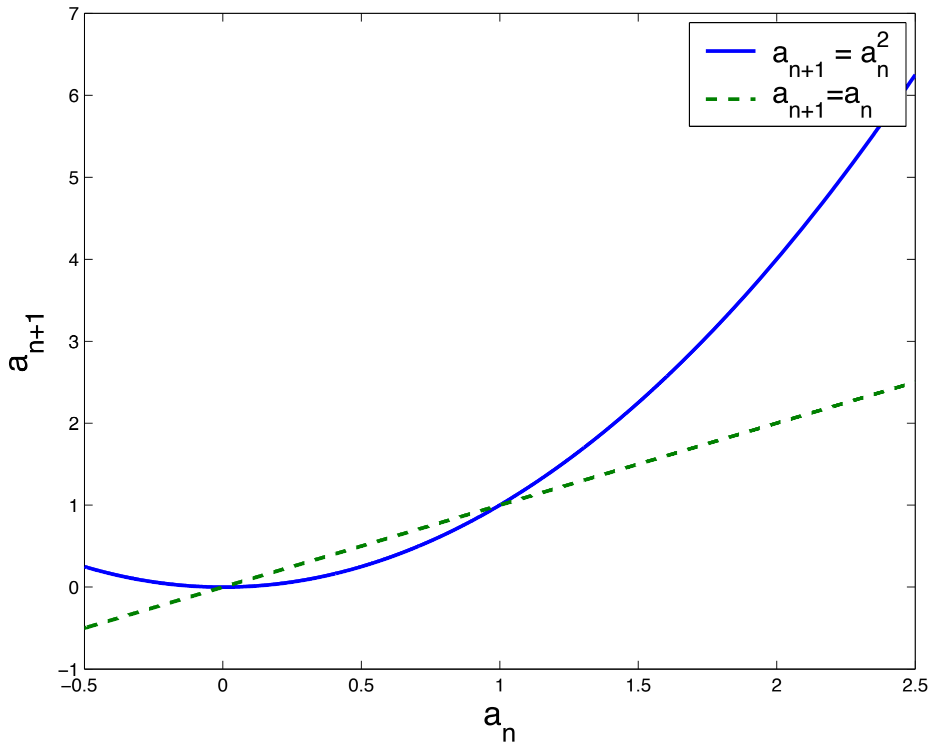


Figure 6: Graph of $a_{n+1} = a_n^2$

PROBLEM SET 2

- We are going to first think about linear updating functions: $f(a_n) = ma_n + b$ for some slope m and y-intercept b.

  1. Discuss the behavior of the sequences generated by $a_{n+1} = 4a_n - 9$ when coupled with the initial conditions $a_1 = 3$, $a_1 = 5$, and $a_1 = 0$ respectively.

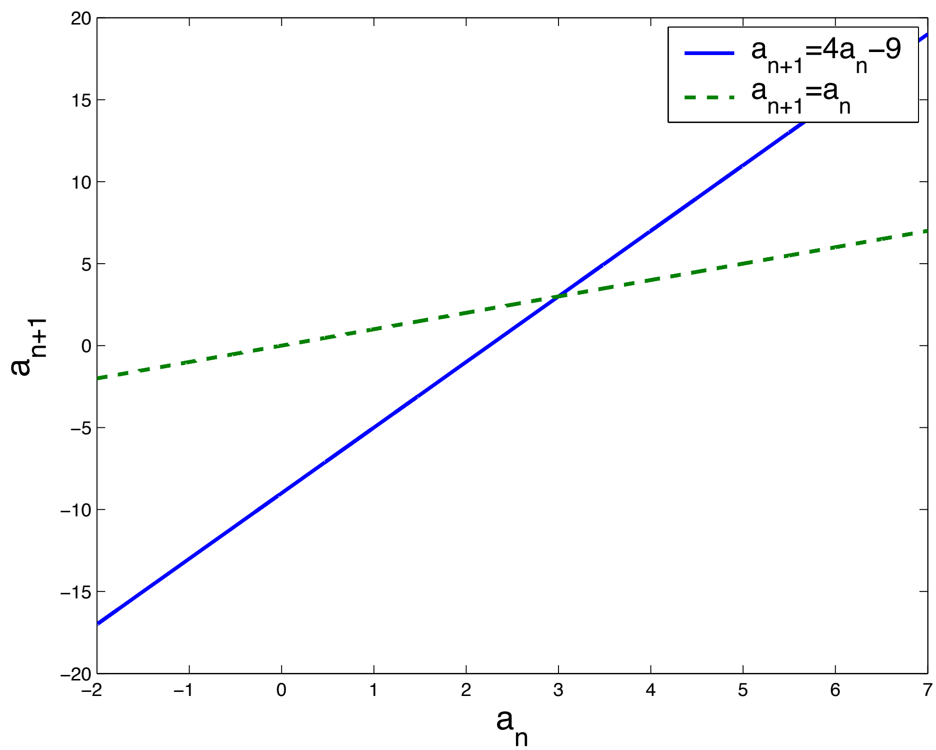


Figure 7: Graph of $a_{n+1} = 4a_n - 9$

2. Discuss the behavior of the sequences generated by $a_{n+1} = \frac{1}{2}a_n + 2$ when coupled with the initial conditions $a_1 = 4$, $a_1 = \frac{9}{2}$, and $a_1 = -1$ respectively.

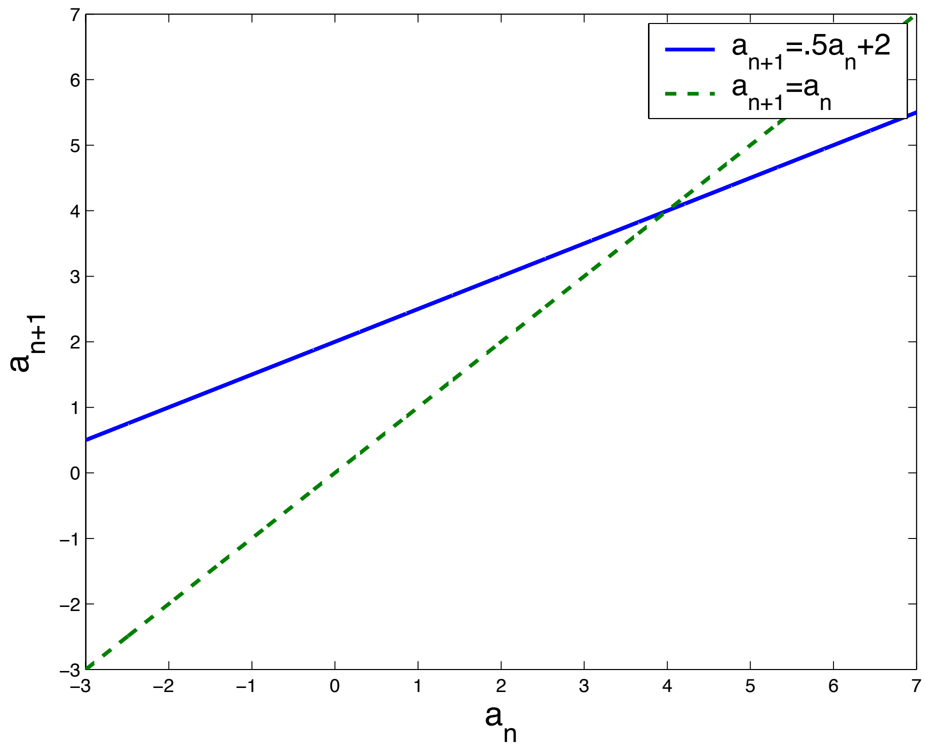


Figure 8: Graph of $a_{n+1} = \frac{1}{2}a_n + 2$

3. Discuss the behavior of the sequences generated by $a_{n+1} = -\frac{1}{2}a_n + 1$ when coupled with the initial conditions $a_1 = 0$, $a_1 = 2$, and $a_1 = 3$ respectively.

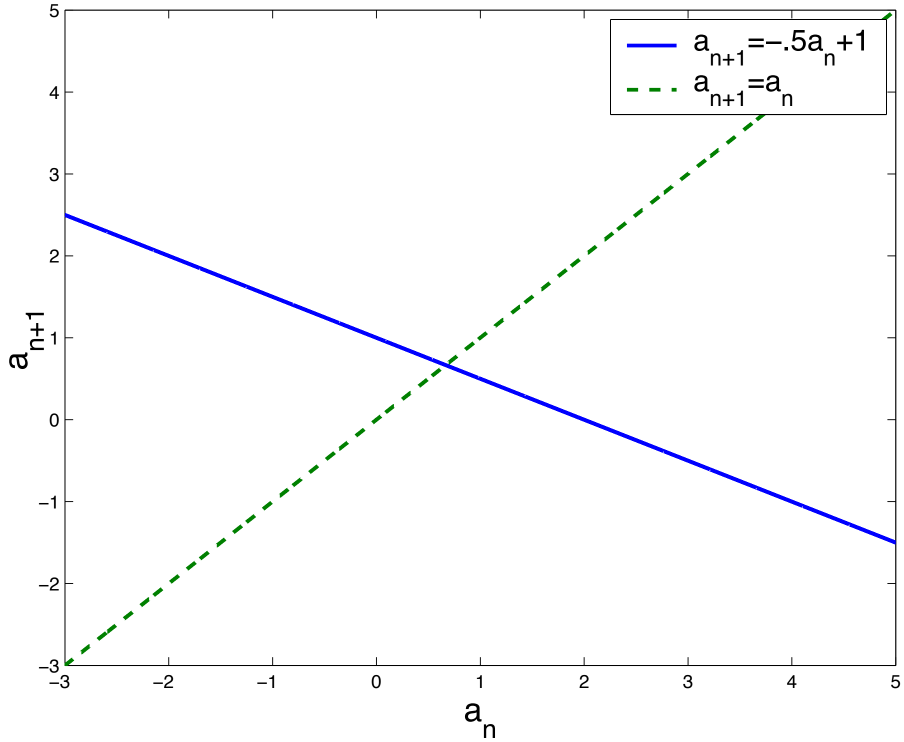


Figure 9: Graph of $a_{n+1} = -\frac{1}{2}a_n + 1$

4. Discuss the behavior of the sequences generated by $a_{n+1} = -3a_n + 6$ when coupled with the initial conditions $a_1 = \frac{1}{2}$, $a_1 = \frac{3}{2}$, and $a_1 = \frac{5}{2}$ respectively.

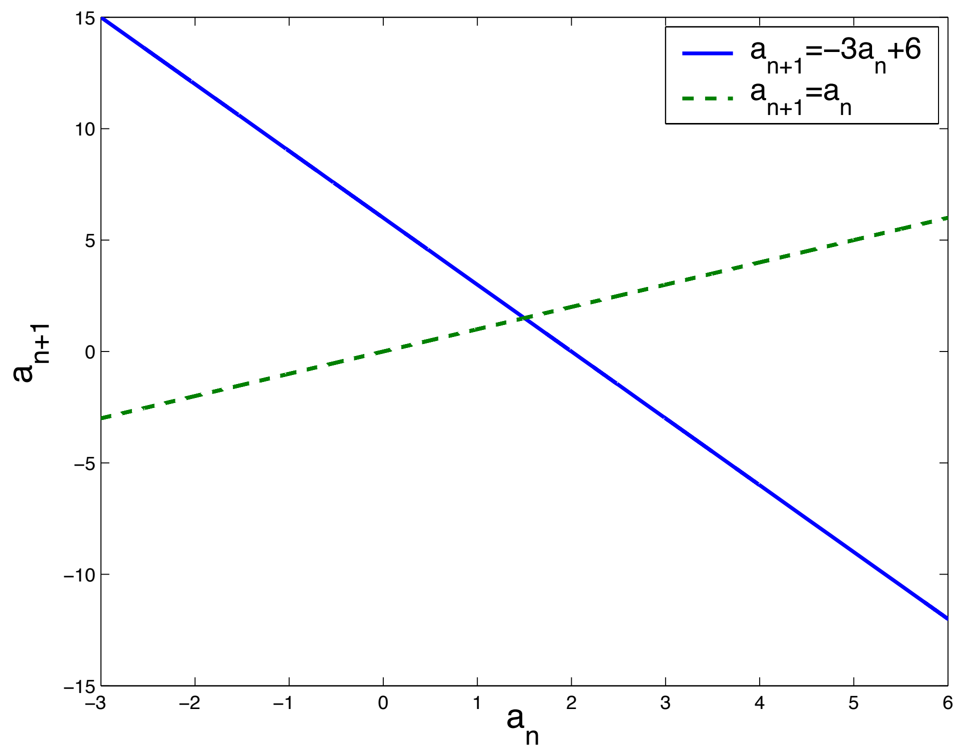


Figure 10: Graph of $a_{n+1} = -3a_n + 6$

5. Discuss the behavior of the sequences generated by $a_{n+1} = a_n + 1$ when coupled with any initial condition.

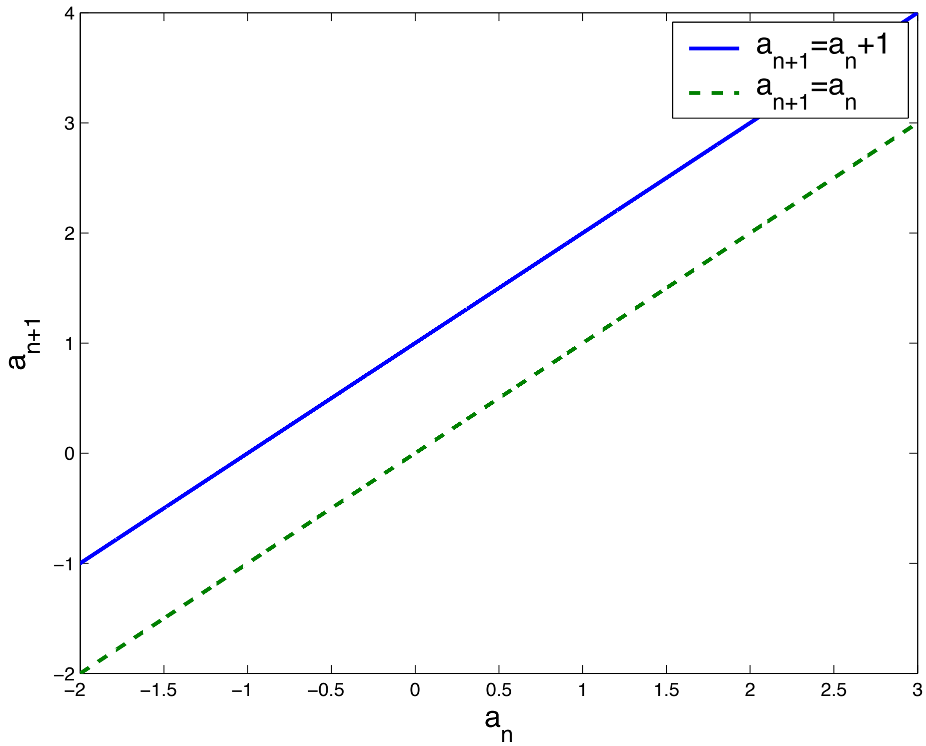


Figure 11: Graph of $a_{n+1} = a_n + 1$

6. Discuss the behavior of the sequences generated by $a_{n+1} = a_n$ when coupled with any initial condition.

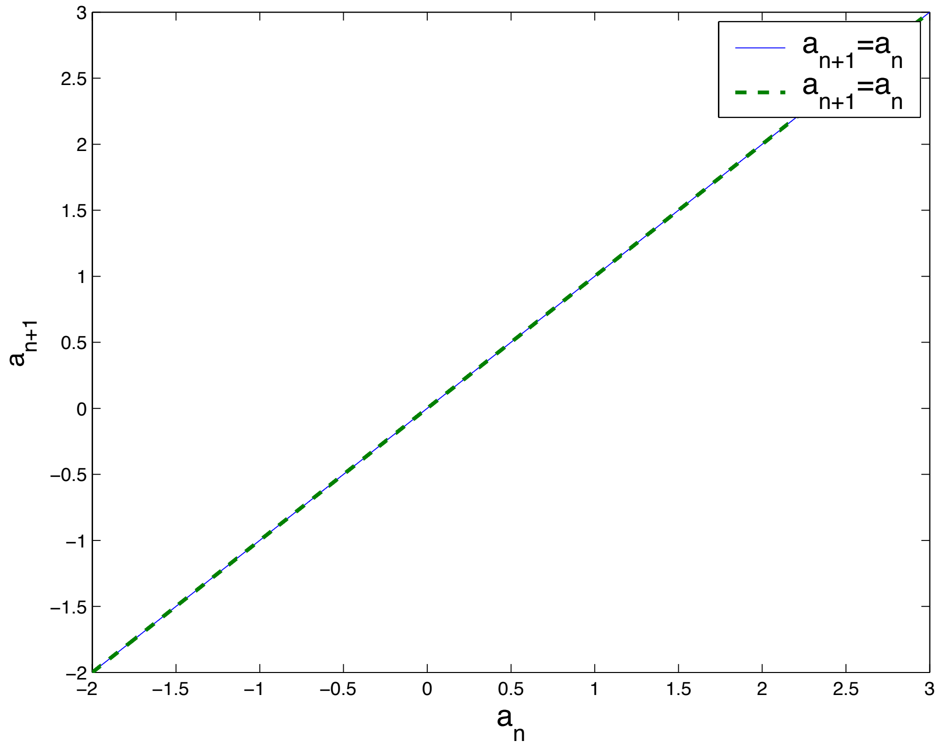


Figure 12: Graph of $a_{n+1} = a_n$

- Now for some non-linear updating functions!

  1. Discuss some possible behaviors of sequences generated by $a_{n+1} = -a_n^2 + 1$. This is more of an exploratory exercise. Pick some initial conditions and see what happens!

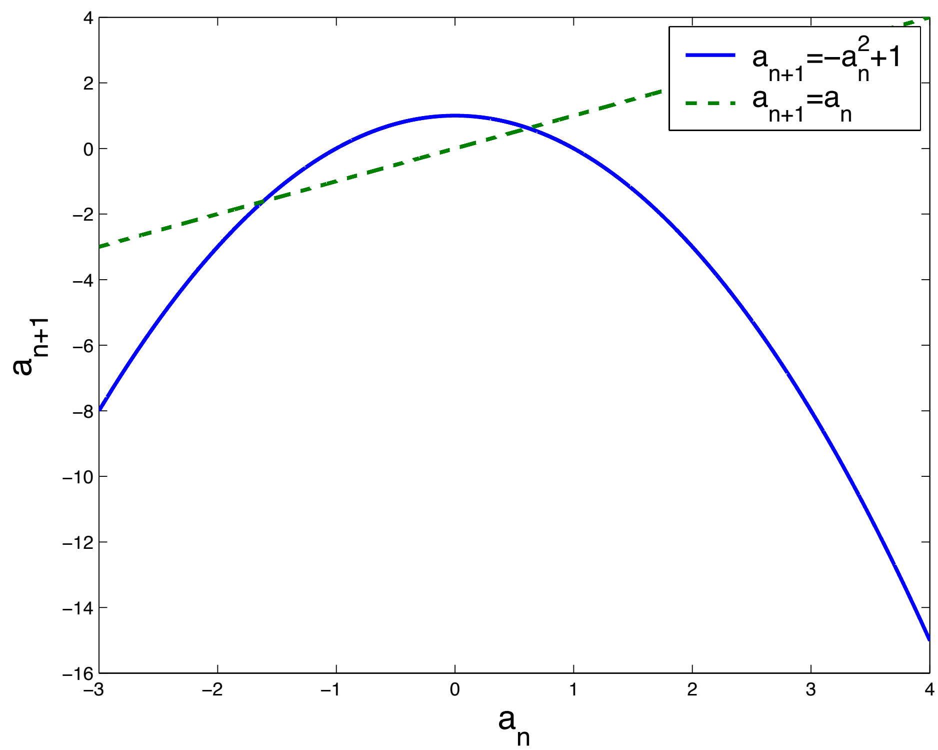


Figure 13: Graph of $a_{n+1} = -a_n^2 + 1$

2. Discuss some possible behaviors of sequences generated by $a_{n+1} = a_n^3$. Again, pick some initial conditions and see what happens.

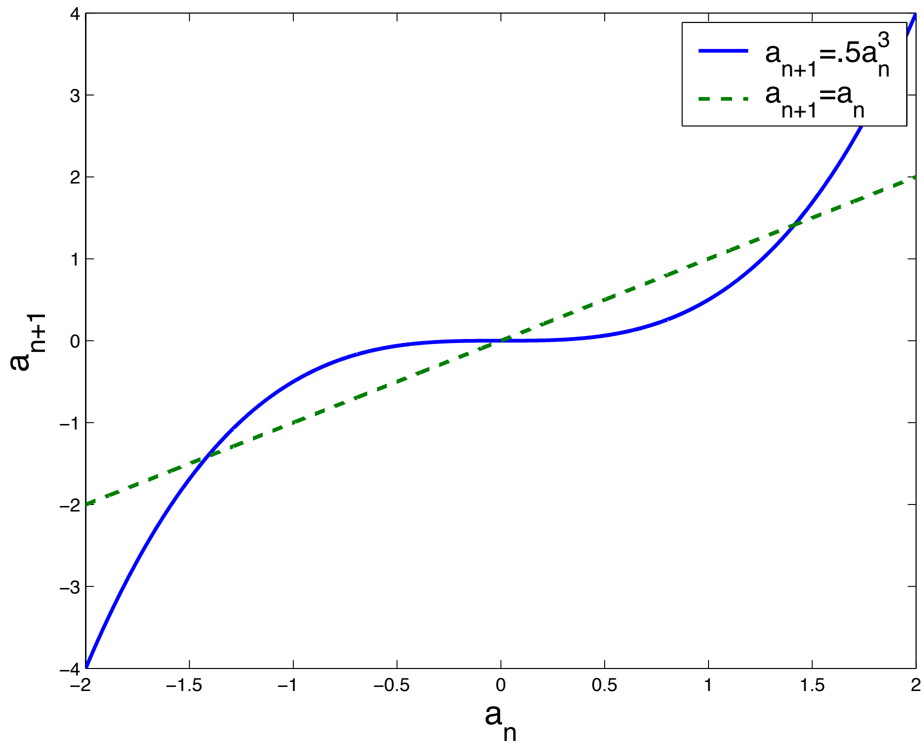


Figure 14: Graph of $a_{n+1} = a_n^3$

3. Discuss some possible behaviors of sequences generated by $a_{n+1} = a_n e^{1-\frac{a_n}{10}}$. Again, pick some initial conditions and see what happens.

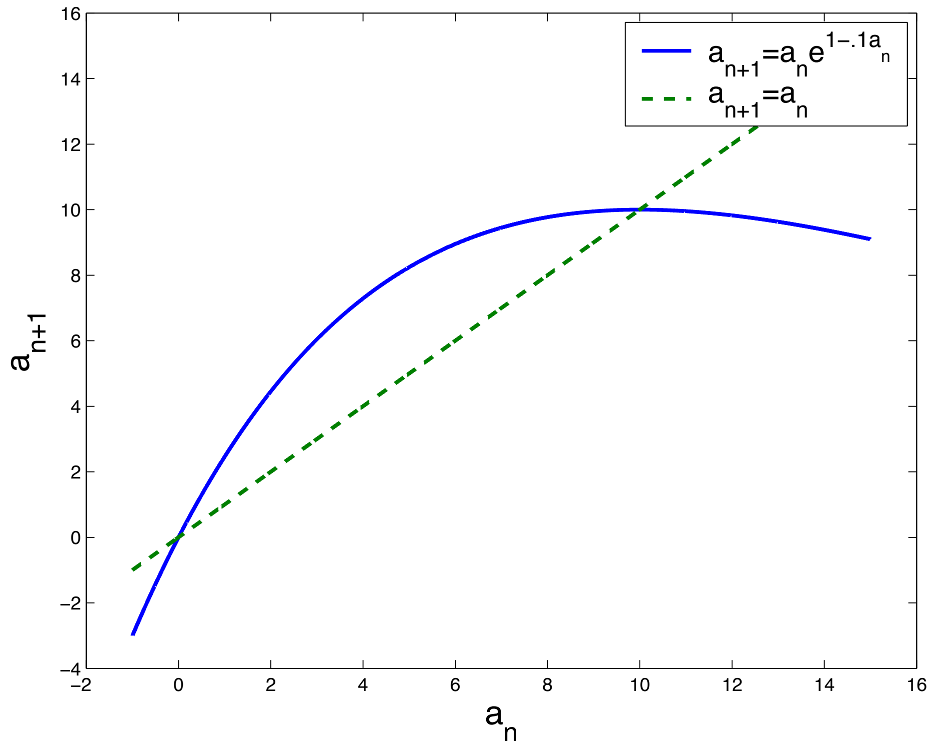


Figure 15: Graph of $a_{n+1} = a_n e^{1-\frac{a_n}{10}}$