PREFACE TO MLADEN BESTVINA'S NOTES ON THE CATALAN NUMBERS:
CLEVER WAYS TO COUNT FINITE SETS

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"Mine is a long and sad tale!" said the Mouse turning to Alice and sighing.
"It is a long tail certainly," said Alice, "but why do you call it sad?"

— *from* Alice in Wonderland

Last time we decide that we should consider two setes S and T to have the same size (or cardinality) if there is a one-to-one onto map f from S to T. We saw that this was consonant with our usual notion of size for finite sets: two finite sets S and T have the same size if and only if they have the same number of elements. Our notion of size can lead to quite interesting phenomena (as we saw last time) for infinite sets, but it seems boring and obvious for finite sets. Not so! That's the point of this session. We'll draw from Chapter 17 of *The Art of Problem Solving, Volume 2* by Rusczyk and Lehoczyk, as well as from the notes of Mladen Bestvina on the Catalan numbers.

Our first example came from a recent contest problem: how many nonnegative integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 \leq 500?$$

Let S denote the set of nonnegative solutions to the inequality. Now let T denote the set of nonnegative integer solutions to the *equality*

$$y_1 + y_2 + y_3 + y_4 + y_5 = 500.$$

The first key step is to notice that there is a one-to-one correspondence between S and T. It takes the solution (x_1, x_2, x_3, x_4) in S to the solution

$$(x_1, x_2, x_3, x_4, 500 - (x_1 + x_2 + x_3, x_4))$$

The second key is to note that we can easily count the number of elements of T using a standard "stars and bars" argument. More formally, let U denote the number of linearly arranged configurations of 500 stars and 4 bars. Then there is a one-to-one correspondence between U and T: a configuration of stars and bars in U corresponds to the solution $(y_1, \ldots, y_5)$ in T where y_1 denotes the number of stars to the left of the first bar; y_2 denotes the number of stars between the first and second bars; y_3 denotes the number of stars between the second and third bars; y_4 denotes the number of stars between the third and fourth bars; and finally y_5 denotes the number of stars to the right of the fourth bar. Now it is easy to see (by a standard interpretation of the binomial coefficient) that there are $\binom{504}{4}$ elements of U. So there are $\binom{504}{4}$ elements of T and hence $\binom{504}{4}$ elements of S. There are other ways to count the elements of S, but using T and U is a particularly elegant way to do so.

The next page contains a few exercises along these lines. After completing them we'll turn to Bestvina's notes on the Catalan numbers.

1. A dog trainer wants to buy 8 dogs all of which are either cocker spaniels, Irish Setters, or Russian Wolfhounds. How many choices does she have?

2. How many solutions solutions are there to $x_1 + x_2 + x_3 \leq 55$ for strictly positve integers x_1, x_2, and x_3.

3. How many ways are there to seat five people in a row of 20 chairs so that no two people sit next to each other?

4. When $(a + b + c + d)^{10}$ is expanded and like terms are combined, how many terms are in the result?