ALGEBRAIC CURVES

A 2003 Math Circle session by Renzo

INTRODUCTION, MOTIVATION AND GOALS

We are going to make a first few exploratory steps in the fascinating but mysterious world of algebraic geometry. Algebraic geometry is a branch of mathematics that studies geometric problems with the help of algebra (like the more geometrical oriented people would say) or algebraic problems with the help of geometry (like the algebraic people would say). Whether one opinion is more correct than the other is a minor point. What's really cool is that there is such a close connection between these apparently unrelated fields of mathematics.

We are going to look in particular in the world of curves, which have the big advantage of being "drawable" and hence a little more intuitive and less scary. Algebraic curves have been studied for a very long time. The ancient Greeks were already pretty handy with them and had figured out a fair amount of things about curves that allowed them to solve famous geometrical problems (such as the trisection of the angle or the doubling of the cube, but this is another story...). Curves are still an extremely interesting object of study nowadays. Even though a LOT of things are known about them, there are many questions that are still open. For example we are still trying to understand the geometry of various moduli spaces of curves (which you can think of as "spaces of curves with given characteristics". For example, all circles in the plane are identified uniquely by the coordinates of the center and by the radius. Hence we can think that there is a whole three dimensional space of circles: you give me three numbers, and I will produce you a corresponding circle; you give me a circle, I'll tell you the three corresponding numbers).

Along the way we'll have a chance to introduce a "new world", where there are no such things as parallel lines. Again, as bizzarre as this may seem, it's an extremely handy things to do. And it's no funky modern invention: the theory of projective geometry was initially developed by architects and painters during the Renaissance, and heavily used for prospectic studies and all sort of applications.

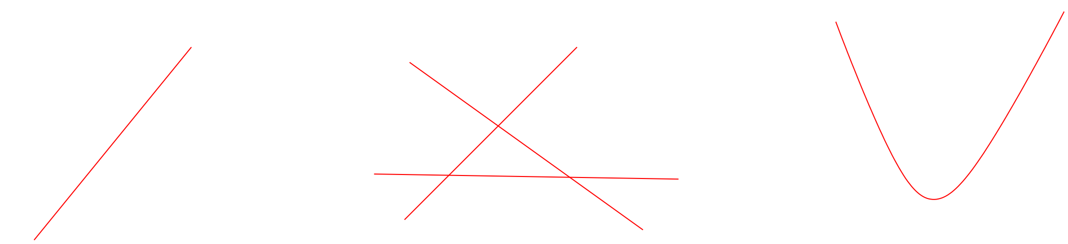
Have I convinced you that we are embarking in a worthwile, even though maybe a little mind quenching adventure? Let's go then!!

DEFINITION: An **Algebraic Curve** consists of the set of points in the plane that satisfy a polynomial equation.

$$C = \{(x, y) \in \mathbb{R}^2 | P(x, y) = 0\}$$

The **Degree** of the curve is just what we expect it to be: the degree of the polynomial defining the curve.

EXAMPLES:



a line — ax+by+c=0

many lines — (what is the equation?)

a parabola — $y=x^2$

Q1: Given two curves C_1 and C_2 with equations $f_1 = 0, f_2 = 0$, what is the equation for the curve $C_1 \cup C_2$?

DEFINITION: A curve C is said **Irreducible** if it does not break up into the union of "simpler" curves.

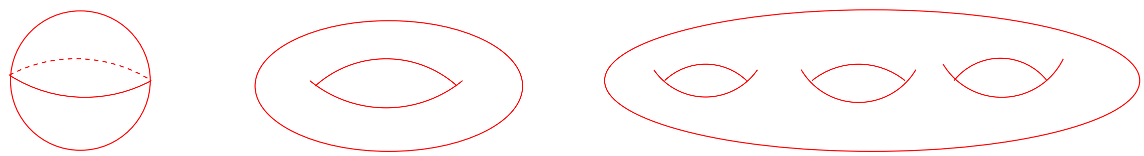
Q2: How does this translate in terms of the equation for C?

Q3: Is the equation for an algebraic curve unique?

PROBLEM...Our "dictionary" doesn't work so well...We want a good correspondence between algebra (polynomial equations) and geometry (curves). A dictionary that uses the words banana, broccoli and balalaika to mean the exact same thing is not all that good, is it?

FIX 1:We are using the **WRONG** numbers. From now on our numbers will be all the complex numbers and not just the real ones! (Alas, we will have to continue to draw curves in the same way because we yet haven't discovered 4-dimensional chalk!)

Notice, though, that now curves will be " objects of complex dimension 1", which means they really are surfaces. Now maybe you remember that we know surfaces pretty well. We could even work out that all complex algebraic curves are orientable, and hence must look like a sphere or a multiholed doughnut (eventually pinched or punctured),



but that's another story....

Q4: What happens now to

$$x^2 + y^2 = 0$$

$$x^2 + y^2 = -1?$$

Q5: Is the fix a complete fix?

If you ponder a little bit you will find out that the answer is "almost". In fact we are still defining the same curve if we multiply the equation by a constant...but that's not such a big deal, and we'll see later on that we'll be able to fix this too by just slightly modifying our "universe".

For now we have estabilished the following bijective correspondence:

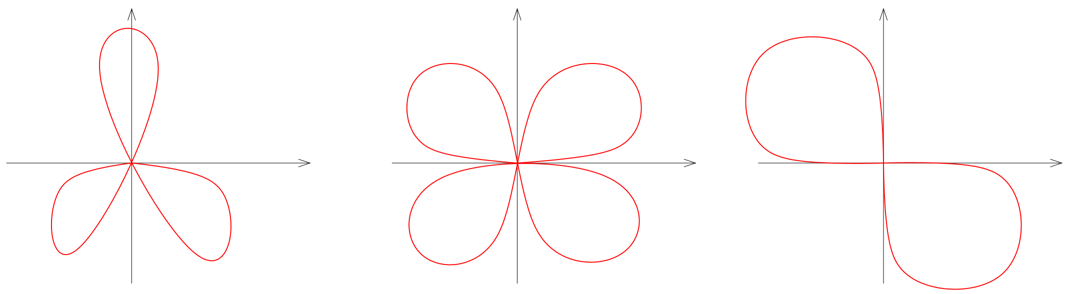
$$\{ \text{ algebraic curves } \} \quad \leftrightarrow \quad \{ \text{ equations up to scaling } \}$$

NOW FOR SOME MORE FUN EXAMPLES:

Cubics: What are the graphs (of the real trace) of:

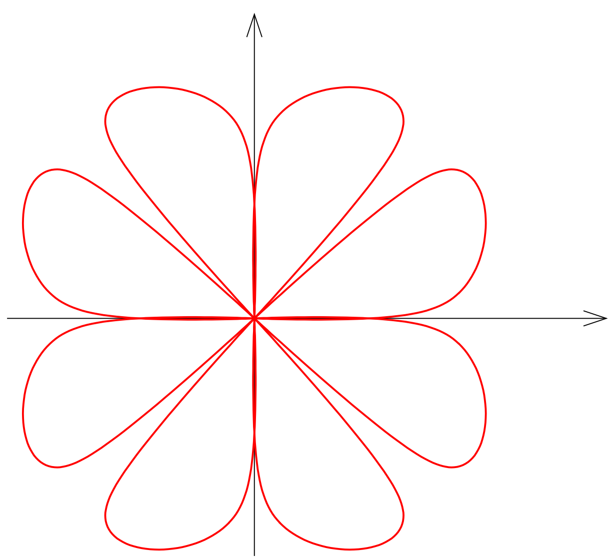
- $y^2 = x^3 - x$;
- $y^2 = x^3$;
- $y^2 = x^3 + x^2$.

Pretty Flowers: Match the graphs with their equations:



- $(x^2 + y^2)^3 - 4x^2y^2 = 0$;
- $(x^2 + y^2)^2 + 4xy = 0$;
- $(x^2 + y^2)^2 - 3x^2y - y^3 = 0$;

More pretty flowers...: Can you produce a sensible guess for the equation of the following figure?

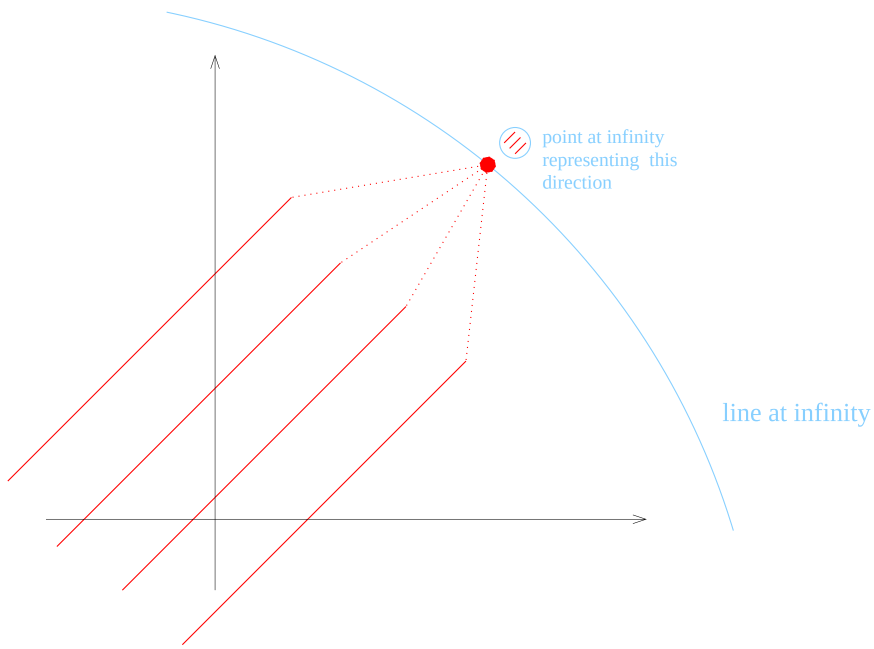


Now that we are starting to get a little more familiar with Algebraic Curves, let's start asking ourselvs interesting questions, and let's start easy to then get harder and harder:

1. How many points of intersection can 2 lines in the plane have?
2. How many points of intersection do a line and a conic have in the plane?
3. How many points of intersection does a line have with a curve C if degree d?
4. How many lines through one point?
5. how many lines through two points?
6. How many points do you need to pin down a unique conic?
7. Will any configuration of the right number of points pin down a unique conic?
8. What's a quick way to determine the equation of the conic through $P_1 = (-1, 0)$, $P_2 = (0, 0)$, $P_3 = (0, 1)$, $P_4 = (-1, 1)$, $P_5 = (\sqrt{2} - \frac{1}{2}, \frac{1}{2})$?
9. How many points pin down a unique curve of degree d?

THE "IDEAL" UNIVERSE : $\mathbb{P}^2_{\mathbb{C}}$

Take the usual plane and "add" to it a new (projective) line, that we will call **line at infinity**. Points of this line represent directions of lines in $\mathbb{C}^2$.



Notice that now every pair of lines meet at exactly one point!!!!

FORMALLY: We add an extra coordinate, so that points in $\mathbb{P}^2_{\mathbb{C}}$ will be identified by three complex numbers

$$(a : b : c).$$

BUT (there's a big but!) we declare two points to be the same point if the coordinates are proportional. So, for example

$$(1 : 2 : 3) = (2 : 4 : 6) = (23 : 46 : 69) = (k : 2k : 3k).$$

Now we can see that

$$\mathbb{P}^2_{\mathbb{C}} = \{(a : b : c), c \neq 0\} \cup \{(a : b : 0)\} = \mathbb{C}^2 \cup l_\infty.$$

What happens to algebraic geometry now? We could a priori just take polynomials in three variables (for example $P(x, y, z) : x^2 + y + 3z + 4 = 0$). But obviously that's completely nuts!!

Q6: Why?

In general I have no hope to make sense of the notion of the "value" of a polynomial at a point of $\mathbb{P}^2_{\mathbb{C}}$. But there are some special polynomials such that at least the zero-set is well defined.

Q7: Can you guess which are they?

DEFINITION: A **Homogoneous** polynomial is a polynomial in which all the monomials have the same degree.

Q8: Gimme some examples.

Q9: Why are homogeneous polynomials the "good ones" for our purposes?

DEFINITION: A **Projective Curve** is the zero set of a homogeneous polynomial. For example:

$$X^2 + 2XY + Y^2 + 3Z^2 = 0.$$

Q9: Can we go back and forth between our previous notion of curves and this current one?

The answer is *homogenize* and *dehomogenize*!!!!
For example:

$$x^2 + y^2 - 1 = 0 \leftrightarrow X^2 + Y^2 - Z^2 = 0,$$

$$y - x^2 = 0 \leftrightarrow YZ - X^2 = 0.$$

Q10: Are homogeneization and dehomogeneization always inverse operations to each other?

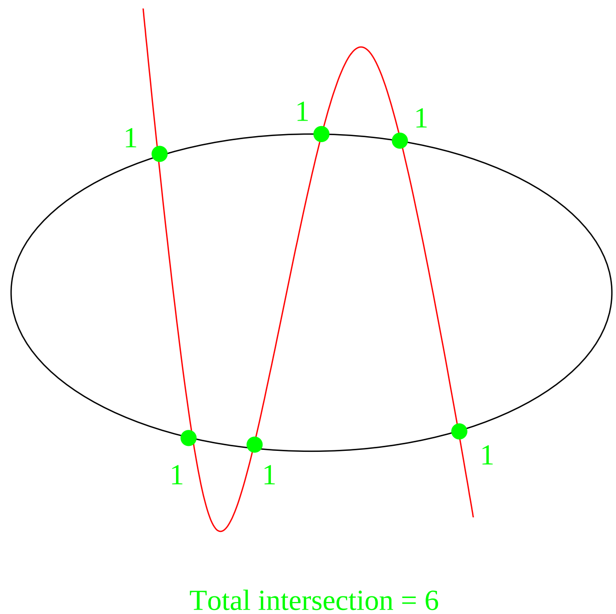
And now for some more problems and interesting stuff to discover!

1. Find algebraically the intersection points in projective space of the following pair of lines:
   - $y = x$;
   - $y = x + 1$.
2. Find algebraically the intersection points in projective space of the following line and hyperbola:
   - $0 = x$;
   - $yx = 1$.
3. Make sense of the statement: "All conics in $\mathbb{P}^2_{\mathbb{C}}$ are potatoes".
4. $\mathbb{P}^2_{\mathbb{C}}$ gives us a real easy way to decide whether a (non projective) conic is a parabola, ellipse or hyperbola. What is it?
5. Decide which of the following conics is a parabola, ellipse or hyperbola!
   - $x^2 + 2xy + 1 + y^2 + 2y = 0$;
   - $x^2 + 3xy + 2y^2 + 2y + 2x + 6 = 0$;
   - $x^2 + xy + y^2 + 4y + 3x - 12 = 0$.

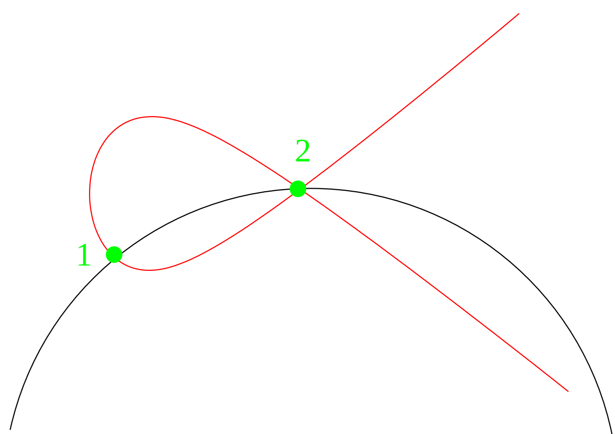
A TINY BIT OF INTERSECTION THEORY

We want to define a notion of intersection of curves that respects the following ideas:

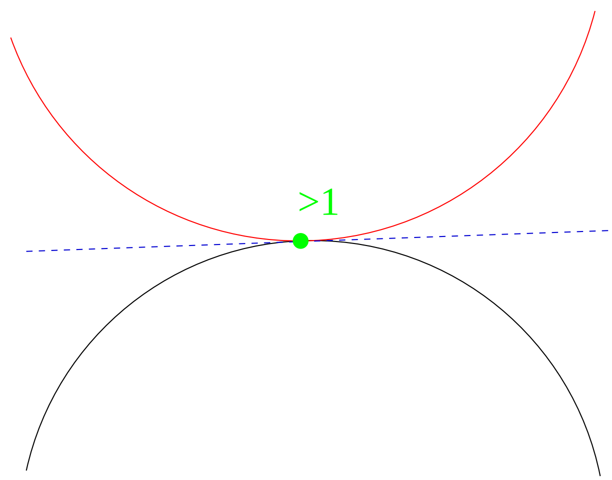
1. If two curves intersect transversally we count exactly the number of points of intersections.



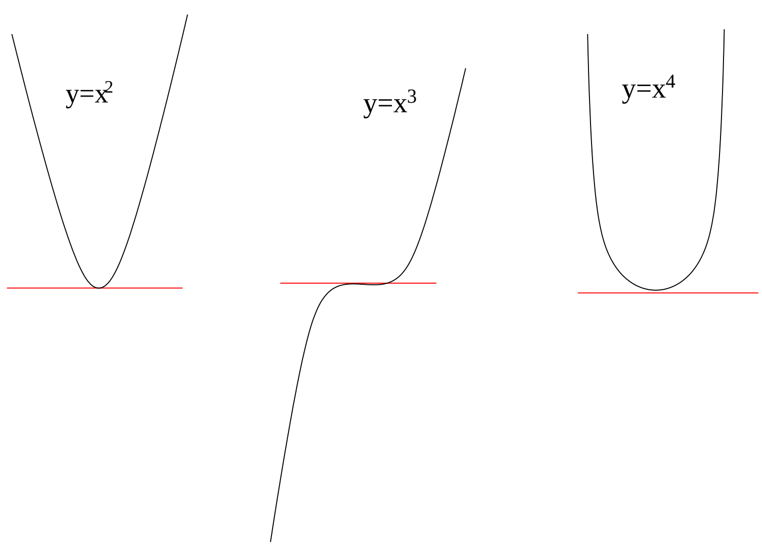
2. If one curve goes through a multiple point of the other curve, but doesn't share a tangent with any "branch", then the point of intersection counts for exactly the multeplicity of the multiple point.



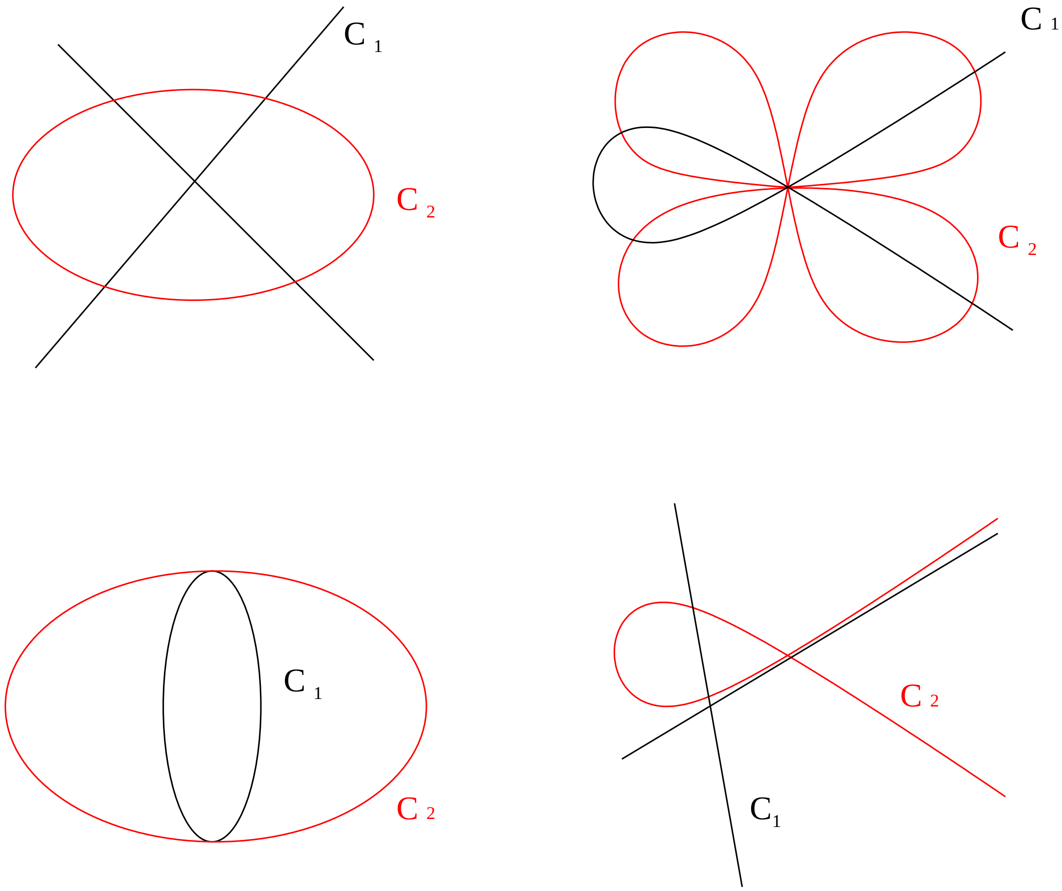
3. If two curves intersect at a point and share the tangent line, then the intersection multeplicity goes up!! (We could make this notion precise, but we won't, it would take just too long!)



Q11: We actually can find a way to make point 3 precise if one of the curves is just a line. Can you think of how?



EXERCISE: Find $|C_1 \cap C_2|$ in the following cases:



Now here comes our big guy. The following theorem is quite straightforward to prove in the case one of the curves is a line (think about why...), but requires some sophistication to be proven in full glory in the following form:

BEZOUT'S THEOREM: If C_1 and C_2 are two projective curves of degree d_1 and d_2 then

$$|C_1 \cap C_2| = d_1 d_2.$$

With the help of Bezout we are finally ready to tackle a bunch of really interesting questions and problems and to prove the conjectures we started out from:

1. If a conic has a double point, then it splits into 2 lines.
2. A conic cannot have a triple point.
3. If a cubic has a triple point, then it's the union of three lines.
4. A cubic with two double points must be the union of a line and a conic.
5. A cubic with three distinct double points must be...?
6. An irreducible quartic cannot have 4 double points.
7. **Pascal's Theorem:** The opposite sides of a hexagon inscribed in a conic meet in collinear points.
8. **Pappus' theorem:** Given two lines l_1, l_2, and chosen randomly three points on each line (call them $P_1, P_2, P_3, Q_1, Q_2, Q_3$), then the points

$$R_1 = \overline{P_2Q_3} \cap \overline{P_3Q_2},$$

$$R_2 = \overline{P_1Q_3} \cap \overline{P_3Q_1},$$

$$R_3 = \overline{P_1Q_2} \cap \overline{P_2Q_1}$$

are collinear.