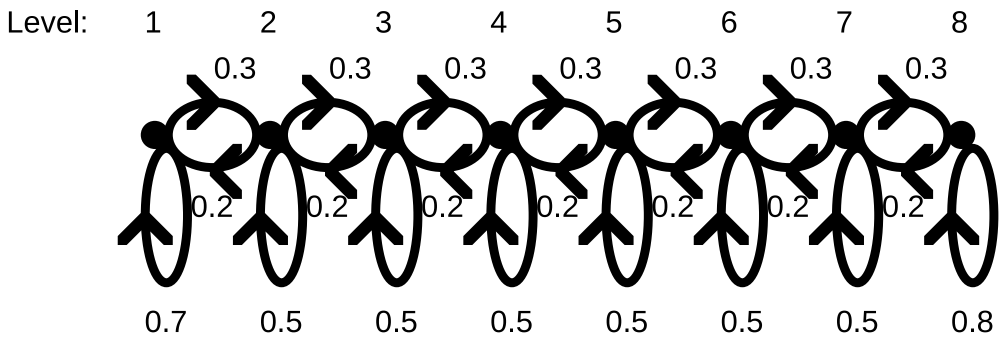
Diffusion Model

Math Circle, October 23, 2002

Here is the directed graph corresponding to a model for the settling of silt in a barrel of water. There are eight levels from the top to the bottom of the barrel. For each level and each time increment, 3/10 of the mud moves down a level (if possible), 2/10 moves up a level (if possible), and the remainder stays in the same level



Here is the transition matrix which in entry i of column j is the fraction of material in level j which moves to level i during a single time increment.

$$A := \begin{bmatrix} .7 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & .5 & .2 & 0 & 0 & 0 & 0 & 0 \\ 0 & .3 & .5 & .2 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & .5 & .2 & 0 & 0 & 0 \\ 0 & 0 & 0 & .3 & .5 & .2 & 0 & 0 \\ 0 & 0 & 0 & 0 & .3 & .5 & .2 & 0 \\ 0 & 0 & 0 & 0 & 0 & .3 & .5 & .2 \\ 0 & 0 & 0 & 0 & 0 & 0 & .3 & .8 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} .55 & .24 & .04 & 0 & 0 & 0 & 0 & 0 \\ .36 & .37 & .20 & .04 & 0 & 0 & 0 & 0 \\ .09 & .30 & .37 & .20 & .04 & 0 & 0 & 0 \\ 0 & .09 & .30 & .37 & .20 & .04 & 0 & 0 \\ 0 & 0 & .09 & .30 & .37 & .20 & .04 & 0 \\ 0 & 0 & 0 & .09 & .30 & .37 & .20 & .04 \\ 0 & 0 & 0 & 0 & .09 & .30 & .37 & .26 \\ 0 & 0 & 0 & 0 & 0 & .09 & .39 & .70 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} .457 & .242 & .068 & .008 & 0 & 0 & 0 & 0 \\ .363 & .317 & .186 & .060 & .008 & 0 & 0 & 0 \\ .153 & .279 & .305 & .186 & .060 & .008 & 0 & 0 \\ .027 & .135 & .279 & .305 & .186 & .060 & .008 & 0 \\ 0 & .027 & .135 & .279 & .305 & .186 & .060 & .008 \\ 0 & 0 & .027 & .135 & .279 & .305 & .186 & .072 \\ 0 & 0 & 0 & .027 & .135 & .279 & .323 & .282 \\ 0 & 0 & 0 & 0 & .027 & .162 & .423 & .638 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} .3925 & .2328 & .0848 & .0176 & .0016 & 0 & 0 & 0 \\ .3492 & .2869 & .1744 & .0696 & .0160 & .0016 & 0 & 0 \\ .1908 & .2616 & .2641 & .1720 & .0696 & .0160 & .0016 & 0 \\ .0594 & .1566 & .2580 & .2641 & .1720 & .0696 & .0160 & .0016 \\ .0081 & .0540 & .1566 & .2580 & .2641 & .1720 & .0696 & .0184 \\ 0 & .0081 & .0540 & .1566 & .2580 & .2641 & .1756 & .0948 \\ 0 & 0 & .0081 & .0540 & .1566 & .2634 & .3019 & .2902 \\ 0 & 0 & 0 & .0081 & .0621 & .2133 & .4353 & .5950 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} .34459 & .22034 & .09424 & .02624 & .00432 & .00032 & 0 & 0 \\ .33051 & .26561 & .16546 & .07448 & .02240 & .00400 & .00032 & 0 \\ .21204 & .24819 & .23597 & .15970 & .07400 & .02240 & .00400 & .00032 \\ .08856 & .16758 & .23955 & .23525 & .15970 & .07400 & .02240 & .00448 \\ .02187 & .07560 & .16650 & .23955 & .23525 & .15970 & .07472 & .02864 \\ .00243 & .02025 & .07560 & .16650 & .23955 & .23633 & .16906 & .11096 \\ 0 & .00243 & .02025 & .07560 & .16812 & .25359 & .29069 & .29254 \\ 0 & 0 & .00243 & .02268 & .09666 & .24966 & .43881 & .56306 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} .30732 & .20736 & .099060 & .033264 & .007504 & .001024 & .000064 & 0 \\ .31104 & .24854 & .15820 & .077052 & .027296 & .006576 & .000960 & .000064 \\ .22289 & .23729 & .21553 & .14924 & .075660 & .027200 & .006576 & .001056 \\ .11227 & .17337 & .22387 & .21344 & .14910 & .075660 & .027344 & .008064 \\ .037989 & .092124 & .17024 & .22365 & .21344 & .14932 & .077892 & .037856 \\ .007776 & .033291 & .091800 & .17024 & .22397 & .21679 & .16508 & .12258 \\ .000729 & .007290 & .033291 & .092286 & .17526 & .24763 & .28382 & .29217 \\ 0 & .000729 & .008019 & .040824 & .12776 & .27581 & .43825 & .53821 \end{bmatrix}$$

$$A^7 = \begin{bmatrix} .27733 & .19486 & .10099 & .038695 & .010712 & .0020320 & .0002368 & .0000128 \\ .29228 & .23394 & .15192 & .078354 & .031031 & .0090352 & .0018144 & .0002432 \\ .22721 & .22788 & .20000 & .14043 & .075839 & .030705 & .0090448 & .0021600 \\ .13060 & .17629 & .21064 & .19623 & .13994 & .075853 & .031224 & .011920 \\ .054230 & .10473 & .17063 & .20990 & .19626 & .14072 & .080166 & .045863 \\ .015430 & .045741 & .10363 & .17067 & .21107 & .20272 & .16268 & .13108 \\ .0026973 & .013778 & .045790 & .10538 & .18038 & .24401 & .27908 & .29050 \\ .0002187 & .0027702 & .016402 & .060346 & .15480 & .29493 & .43576 & .51822 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} .25259 & .18319 & .10107 & .042758 & .013705 & .0032295 & .00052864 & .00005760 \\ .27479 & .22100 & .14627 & .078871 & .033897 & .011269 & .0027872 & .00055744 \\ .22741 & .21939 & .18770 & .13298 & .075216 & .033233 & .011311 & .0035369 \\ .14431 & .17746 & .19946 & .18222 & .13198 & .075280 & .034357 & .015781 \\ .069379 & .11440 & .16924 & .19796 & .18232 & .13365 & .081985 & .052724 \\ .024524 & .057045 & .11216 & .16938 & .20049 & .19237 & .16121 & .13740 \\ .0060216 & .021165 & .057264 & .11596 & .18447 & .24181 & .27549 & .28822 \\ .00098415 & .0063496 & .026859 & .079890 & .17794 & .30915 & .43233 & .50173 \end{bmatrix}$$

$$A^9 = \begin{bmatrix} .23177 & .17243 & .10000 & .045705 & .016373 & .0045144 & .00092750 & .00015181 \\ .25865 & .20934 & .14100 & .078859 & .036104 & .013250 & .0038146 & .0010034 \\ .22501 & .21149 & .17762 & .12660 & .074174 & .035053 & .013363 & .0050917 \\ .15426 & .17743 & .18989 & .17059 & .12501 & .074343 & .036969 & .019496 \\ .082885 & .12185 & .16689 & .18753 & .17085 & .12789 & .083540 & .058576 \\ .034281 & .067076 & .11830 & .16727 & .19184 & .18464 & .16030 & .14216 \\ .010565 & .028966 & .067653 & .12477 & .18797 & .24045 & .27258 & .28568 \\ .0025938 & .011429 & .038667 & .098699 & .19770 & .31987 & .42852 & .48785 \end{bmatrix}$$

$$A^{10} = \begin{bmatrix} .21397 & .16257 & .098199 & .047765 & .018682 & .0058100 & .0014122 & .00030695 \\ .24385 & .19870 & .13602 & .078459 & .037799 & .014990 & .0048584 & .0015656 \\ .22095 & .20403 & .16908 & .12107 & .072922 & .036369 & .015219 & .0067460 \\ .16121 & .17653 & .18161 & .16078 & .11893 & .073266 & .039202 & .022991 \\ .094574 & .12757 & .16407 & .17840 & .16130 & .12318 & .084919 & .063569 \\ .044118 & .075885 & .12274 & .16484 & .18476 & .17878 & .15973 & .14579 \\ .016085 & .036891 & .077052 & .13231 & .19107 & .23959 & .27008 & .28305 \\ .0052445 & .017833 & .051229 & .11639 & .21455 & .32803 & .42458 & .47599 \end{bmatrix}$$

Can you answer the following questions, assuming our time increment is 1 hour?

(1) If 100 grams of silt/dye started at level 1 at time 0, how many grams are there in each level 10 hours later?

Answer: the amount in level i is the i^{th} entry in the first column of A^{10}, times 100. For example, there will be about 21 grams in level 1.

(2) If there were 100 grams in level 1 and 100 grams in level 3 at time zero (and no grams in each other level), how are the 200 grams distributed 10 hours later?

Answer: Add 100 times the entries in the first column to 100 times the entries in the third column, of A^{10}.

(3) Suppose that at time zero there are x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 grams in each of the 8 levels. Is there a way you could have your calculator compute the resulting amounts in each level after 10 hours? Can you explain why?

Answer: multiply the A^{10} matrix by a very skinny matrix (with 1 column and 8 rows) which has the x_i values down the rows. (This kind of skinny matrix is also called a vector.)

Here are some very large powers of the incidence matrix:

$$A^{1000} = \begin{bmatrix} .020322 & .020322 & .020322 & .020323 & .020322 & .020322 & .020322 & .020322 \\ .030482 & .030482 & .030484 & .030484 & .030484 & .030482 & .030484 & .030482 \\ .045725 & .045725 & .045727 & .045727 & .045726 & .045725 & .045726 & .045725 \\ .068585 & .068585 & .068586 & .068587 & .068586 & .068585 & .068586 & .068585 \\ .10287 & .10287 & .10288 & .10288 & .10288 & .10287 & .10288 & .10287 \\ .15431 & .15431 & .15432 & .15432 & .15432 & .15431 & .15432 & .15431 \\ .23147 & .23147 & .23148 & .23148 & .23148 & .23147 & .23148 & .23147 \\ .34719 & .34719 & .34720 & .34720 & .34719 & .34719 & .34719 & .34719 \end{bmatrix}$$

$$A^{1001} = \begin{bmatrix} .020323 & .020323 & .020322 & .020322 & .020322 & .020322 & .020322 & .020322 \\ .030484 & .030484 & .030484 & .030484 & .030484 & .030484 & .030483 & .030484 \\ .045727 & .045727 & .045727 & .045726 & .045726 & .045726 & .045725 & .045726 \\ .068587 & .068589 & .068586 & .068586 & .068586 & .068586 & .068586 & .068586 \\ .10288 & .10288 & .10288 & .10288 & .10288 & .10288 & .10288 & .10288 \\ .15432 & .15433 & .15432 & .15432 & .15432 & .15432 & .15432 & .15432 \\ .23148 & .23148 & .23148 & .23148 & .23148 & .23148 & .23147 & .23148 \\ .34720 & .34721 & .34720 & .34719 & .34719 & .34719 & .34719 & .34719 \end{bmatrix}$$

(4) If you started with 100 grams of silt, distributed in any way you wanted among the 8 levels, how will the silt be distributed after a very very long time?

Answer: The matrix stabilizes after a long time, i.e. continued multiplying by A leaves it unchanged. Since each column is the same in the stabilized matrix, it doesn't matter where the silt/dye started, it all eventually gets distributed in the same way. There will be about 2 grams in level 1, 3 in level 2, 4.5 in level 3, 6.9 in level 4, 10 in level 5, 15.4 in level 6, 23 in level 7 and 34.7 in level 8!