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The Euler Characteristic. Part I

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1 Background

Recall our definition of a compact and connected surface without boundary. By *compact*, we meant a surface that does not go on forever. A sphere is a compact curface; an infinite sheet of paper is not. By *connected* we meant that the surface should consists on only one piece. A surface without boundary has no edges. A Möbius band has an edge and therefore has a boundary.

Example. Sphere, torus, projective plane, Klein bottle

We also came up with a way of gluing two surfaces together, calling the new surface the *connected sum* of the two surfaces. To form the connected sum, start by cutting out a small hole from each surface and glue them together along the edges of the two holes.

If S and T are the two surfaces, we denote the connected sum as $S\#T$. This turned out to be a useful notion to define, as we were able to figure out that *any* compact and connected surface is descibed by the following.

Theorem (Brahana, 1922). *Suppose that S is a surface that is compact, connected and without boundary. Then S is either:*

1. *A sphere,*
2. *a connected sum of some number of tori, or*
3. *a connected sum of some number of projective planes.*

While very beautiful, the theorem is a little unsatisfactory. Although we know that every compact connected surface without boundary looks like one of the above, it is often very hard to decide precisely which one.

Example. Which surface does the following represent?

Picture of a quotient rectangle.

Goal: We would like to find a reasonably quick way of figuring this out.

2 Polyhedra

Consider a polyhedron, like one of the figures below.

Problem 2.1. For each of these figures, compute the following number:

# of Faces - # of Edges + # of Vertices

This number turns out to be very important (as we will see) and deserves a name. It is called the *Euler characteristic*. For a polyhedron P, we will write $\chi(P)$ for its Euler characteristic.

Problem 2.2. What is the Euler characteristic of this complicated figure (called a stellated icosahedron)?

Problem 2.3. All of the above figures can be stretched to make which compact, connected surface?

The above observation allows us to define the Euler characteristic for this surface. One of our goals will be to define an Euler characteristic for any compact, connected surface.

3 The Torus and the Non-Orientable Surfaces

First, let's try to figure out what the Euler characteristic of a torus should be. To do that we need to find a "polyhedron" that can be deformed into a torus.

Problem 3.1. Compute # of Faces - # of Edges + # of Vertices for a polyhedron that can be deformed into a torus.

Problem 3.2. Compute # of Faces - # of Edges + # of Vertices for a polyhedron that can be deformed into a Klein bottle.

The second problem is much more difficult than the first. It is much harder to visualize the Klein bottle than it is to visualize a torus and consequently, it is much harder to find a polyhedron that looks like a Klein bottle. To deal with this, let's develop some technology. First of all, it is a lot easier to deal with polyhedra that are made up entirely of "triangles". This motivates the following definition.

Definition. A *triangle* is three points (called vertices) connected by three distinct line segments (called edges) together with the interior (called the face of the triangle).

DEFINITION. A *triangulation* of a compact surface consists of a set of triangles drawn on the surface with the following properties:

1. The entire surface is covered by these triangles,
2. any two distinct triangles are either
   - completely disjoint,
   - have only a single vertex in common,
   - have only one entire edge in common.

EXAMPLE. The following configurations of triangles cannot occur within any triangulation(Why?):

PROBLEM 3.3. Construct several triangulations of a sphere and a torus.

PROBLEM 3.4. Construct triangulations of a sphere and torus with using the fewest number of triangles.

PROBLEM 3.5. Now construct a triangulation of a Klein bottle.

Once we've triangulated a surface, we are ready to compute the value of the expression # of Faces - # of Edges + # of Vertices. Fill in the following table:

Surface	# of Faces - # of Edges + # of Vertices
Sphere	
Torus	
Klein Bottle	
Projective Plane	
Two-holed torus	

DEFINITION 3.6. The *Euler Characteristic* of a surface S is the value of

$$\chi(S) = \# \text{ of Faces - } \# \text{ of Edges + } \# \text{ of Vertices}$$

for any triangulation of S.

What is the problem with this definition?

Mind Bender: It has been known for more than 2000 years that there are only five regular polyhedra: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Prove this, considering subdivisions of the sphere into n-gons such that exactly m edges meet at each vertex. (m is fixed, and both m and n are greater than 3) Hint: Use the fact that $\chi(sphere) = 2$.