SALT LAKE MATH CIRCLE, April 2002

The Euler Characteristic. Part II

Thomas Pietraho, University of Utah

1 Compact Surfaces

Last time, we were able to compute the Euler characteristic of a sphere, torus, and Klein bottle. We found that $\chi(sphere) = 2$, $\chi(torus) = 0$, and $\chi(Klein\ bottle) = 0$. We'd like to know the Euler characteristic of any compact surface.

Problem 1.1. Find a formula for the Euler characteristic of the connected sum of two compact surfaces S and T in terms of $\chi(S)$ and $\chi(T)$.

Problem 1.2. The Klein bottle is the connected sum of two projective planes. Using the result of the previous problem, find the Euler characteristic of a projective plane.

Problem 1.3. What is the Euler characteristic of a torus with k holes? What is the Euler characteristic of a connected sum of l projective planes? Now we can write down the Euler characteristic for any compact surface.

Problem 1.4. The connected sum of a four holed torus and three projective planes is a perfectly legitimate compact surface. How does it fit into our classification scheme?

2 Regular Polyhedra

Definition. A polyhedron is *regular* if

- each face is an n-gon, where n is the same for all the faces and is at least 3, and
- each vertex is the intersection of exactly m edges, where m is again the same for all the vertices and at least 3.

The shorthand notation for a regular polyhedron whose faces are n-gons and whose vertices are the intersection of m edges is (n, m).

Problem 2.1. It has been known for more than 2000 years that there are only five regular polyhedra shaped like a sphere: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Prove this, considering subdivisions of the sphere into n-gons such that exactly m edges meet at each vertex. (m is fixed, and both m and n are greater than 3) Hint: Use the fact that $\chi(sphere) = 2$.

Problem 2.2. What are the regular polyhedra shaped like a torus? How many of them are there?

3 Triangulations

Definition. A *triangulation* of a compact surface is a polyhedron that looks like the compact surface and all of whose faces are triangles.

For instance, the icosahedron is a triangulation of a sphere. Triangulations are mathematically simpler to deal with, and are instrumental in the proof of Euler's original observation. One interesting question is "What is the smallest number of faces one can "triangulate" a surface with?" This sequence of excercises is designed to find an answer.

PROBLEM 3.1. Let v, e, and f be the number of vertices, edges, and faces of some triangulation. Let χ be its Euler characteristic. Show that:

- $3f = 2e$
- $e = 3(v - \chi)$
- $v \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi})$ Hint: What is the maximum value of χ for any compact surface?

PROBLEM 3.2. Using the results of the previous problem, what is the minimum number of triangles necessary to triangulate a:

- sphere,
- projective plane,
- a torus.

Finally, construct an example of such a minimal triangulation.