Fractals, Part II

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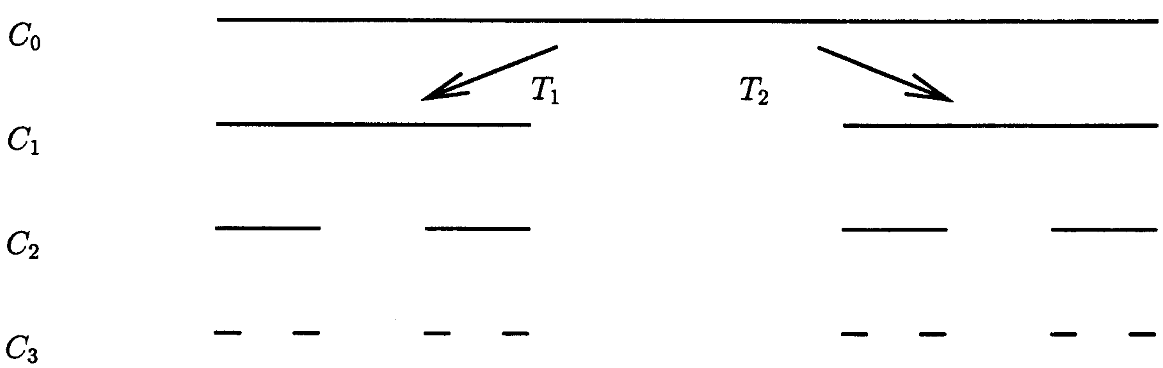
Last week we defined a fractal as an object which can be written as a finite number of copies of itself, each of which is shrunken in some way. Then we recreated some of the classical fractals (like the Cantor set, the Sierpinski triangle and the von Koch snowflake) and discussed some of their properties.

1 The Cantor set and Sierpinski's triangle revisited

Last week we created the Cantor set C by starting with the unit interval $[0, 1]$ and successively deleting the middle third of any intervals we had. However, there is another way create C. Again, we will start with the $C_0 = [0, 1]$. This time, we will create C_1 by

- rescaling C_0 by a factor of $1/3$,
- rescaling C_0 by a factor of $1/3$ and translating it to the right by $2/3$,
- and letting C_1 be the union of these two sets.

Below is a picture of this process.



In general we have the following operation we can apply to any set $A \subset \mathbb{R}$:

- rescale A by a factor of $1/3$
- rescale A by a factor of $1/3$ and translate to the right by $2/3$
- replace A with the union of these two sets.

1. Show that this recipe creates the same $C_1, C_2, C_3, \ldots$ as the deletion process we talked about last week.

2. What happens if you start with a point (say the point 0) instead of the unit interval $[0, 1]$? Draw the first several iterations of this shrinking and translating process starting with 0.

3. What happens to the Cantor set if you apply a step in this recipe?

4. There are two transformations T_1, T_2 from the real line $\mathbb{R}$ to itself in this recipe. Namely, T_1 rescales by a factor of $1/3$ while T_2 rescales by a factor of $1/3$ and translates to the right by $2/3$. Can you write these transformations in terms of a real coordinate x?

5. Recall that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Given two numbers x and $\bar{x}$, what can you say about $|T_1(x) - T_1(\bar{x})|$ as compared to $|x - \bar{x}|$? How about $|T_2(x) - T_2(\bar{x})|$ compared to $|x - \bar{x}|$?

You should have found that the object you get when you use this recipe starting with a point looks a lot like the Cantor set we found before. Indeed, you can start with any nice bounded set and obtain the Cantor set with this recipe. The two transformations are

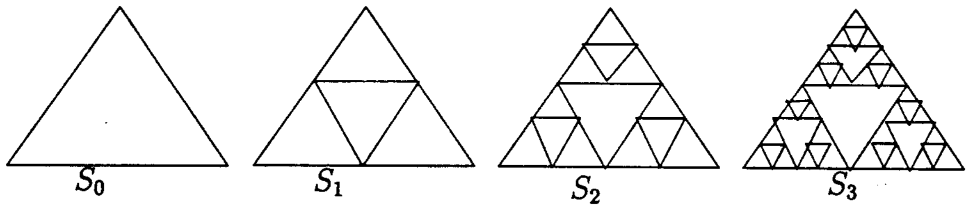
$$T_1(x) = \frac{1}{3}x \qquad T_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Moreover,

$$|T_1(x) - T_1(\bar{x})| = \frac{1}{3}|x - \bar{x}| \qquad |T_2(x) - T_2(\bar{x})| = \frac{1}{3}|x - \bar{x}|.$$

The important thing is that each of the transformations T_1 and T_2 shrink distances by some fixed amount.

We can create Sierpinski's triangle S with a similar shrinking and translating process. Here is a picture of the first several iterations in creating S.



This time we will have three transformations instead of two. All three transformations T_1, T_2, T_3 map the plane $\mathbb{R}^2$ to itself. The first transformation is a uniform rescaling by a factor of $1/2$. We can write this transformation as

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

The second transformation is a uniform rescaling by a factor of $1/2$ followed by a translation to the right by $1/2$. We can write the second transformation as

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

The third transformation is a uniform rescaling by a factor of 1/2 followed by a translation to the right by 1/4 and up by 1/2. We can write the third transformation as

$$T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} = x\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}.$$

Again, our recipe is to replace a set A with the union $T_1(A) \cup T_2(A) \cup T_2(A)$. You can think of these transformations in the following way. For instance, T_2 takes the x axis and first rescales it by a factor of 1/2 and then translates it to the right by 1/2. Similarly, T_2 takes the y axis and first rescales it by a factor of 1/2 and then translates it to the right by 1/2.

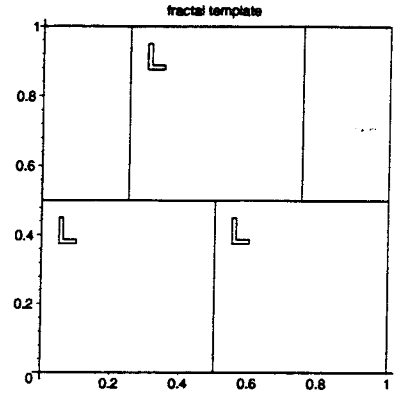
1. Draw the first several iterations of this recipe if you start with an equilateral triangle, a unit square and a unit circle. Do you notice a pattern?
2. What happens to the Sierpinski triangle when you apply a step in this recipe?
3. Given two points in the plane $P = \begin{bmatrix} x \\ y \end{bmatrix}$ and $Q = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$, what can you say about the distance between $T_1(P)$ and $T_1(Q)$ as compared to the distance between P and Q? Answer the same question for T_2 and T_3.

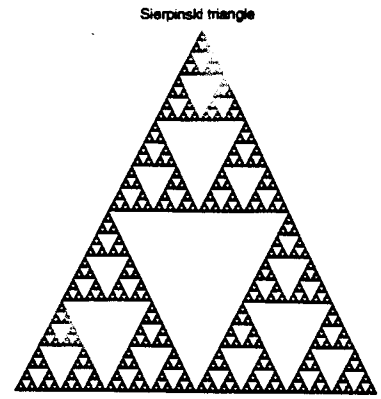
You should have found that you always obtain Sierpinski's triangle with this recipe, no matter if you start with a triangle, square, circle or some other bounded shape. Moreover, the transformations T_1, T_2, T_3 again decrease distance (this time by a factor of 1/2).

2 Some transformations of the plane: rescalings and translations

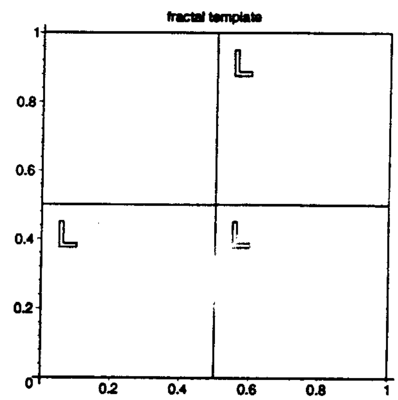
Let's take a closer look at the transformations of the plane we are using. Recall that the unit square is the set of points $\begin{bmatrix} x \\ y \end{bmatrix}$ in the plane such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. We will denote the unit square by $\Box$. In all the pictures on the left below, the large square is the unit square. The smaller rectangles are the images of the unit square under T_1, T_2, T_3; $T_1(\Box)$ is always the lower left hand image, $T_2(\Box)$ is the lower right hand image, and $T_3(\Box)$ is the top image.

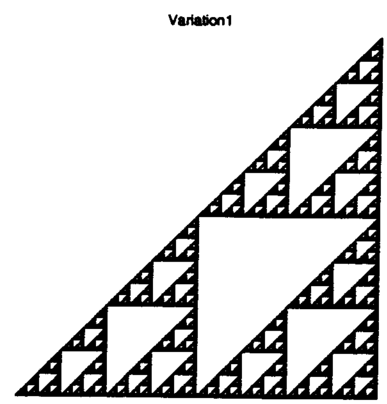
Below is a picture of the image of the unit square $\Box$ under all three of the transformations T_1, T_2, T_3 we used to build Sierpinki's triangle.



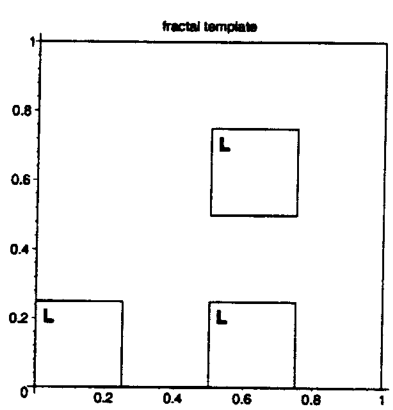


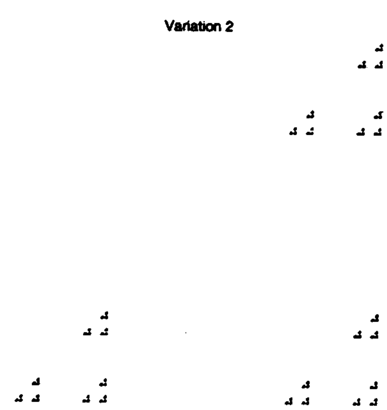
However, we could have made a mistake and translated the top square too far to the right. Then the picture of the image of the unit square under all three transformations would look like this picture.



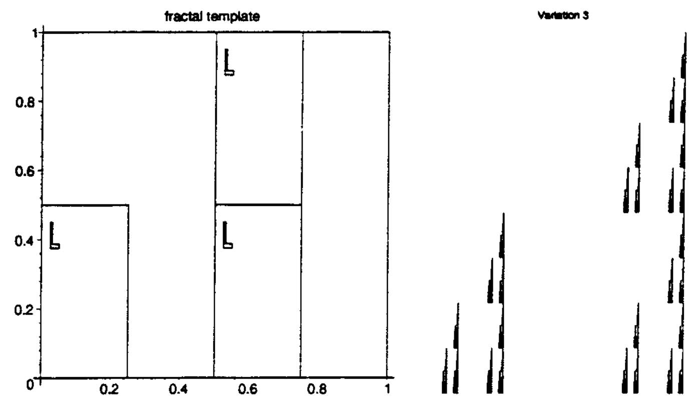


Additionally, we could rescale everything by a factor of 1/4 instead of a factor of 1/2. Then the picture of the image of the unit square under all three transformations would look like this picture.



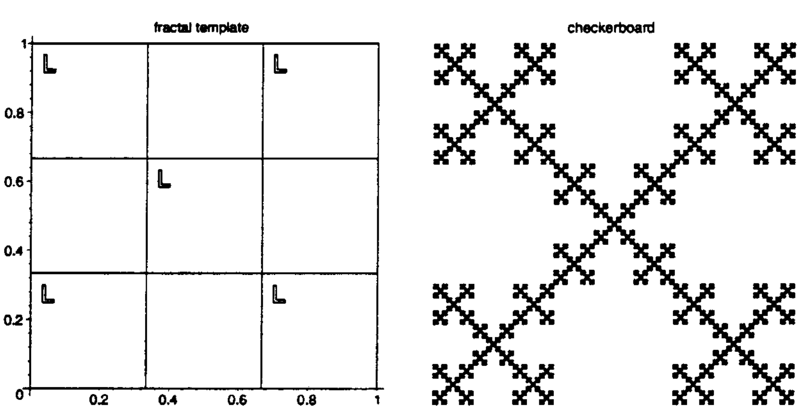


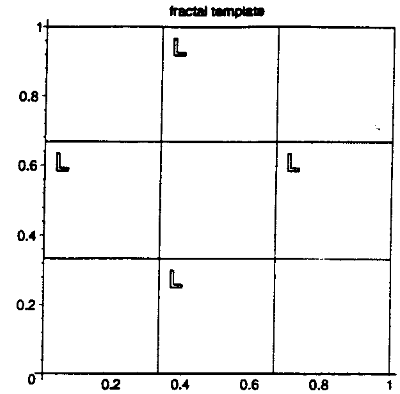
Even worse, the rescalings could be by a factor of 1/2 in the vertical direction and by a factor of 1/4 in the horizontal directions. Then the picture of the image of the unit square under all three transformations would look like this picture.

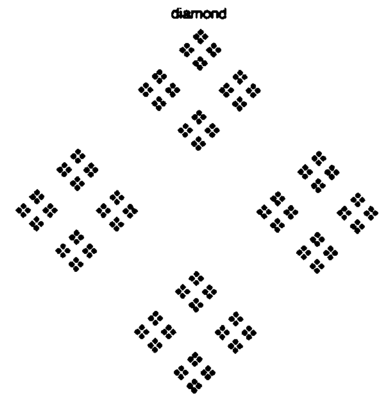


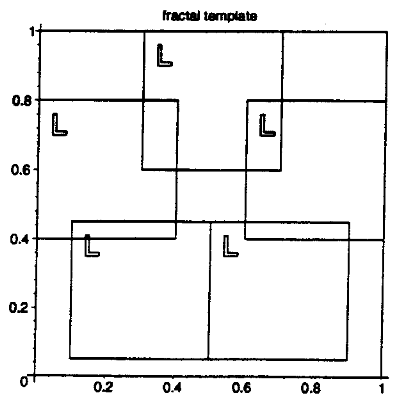
1. Can you find formulas for the new transformations in each of these cases?
2. Can you make up a few similar transformations of your own and write down their formulas?
3. Can you write down the formula for a general translation, say to the right by e and up by f?
4. Can you write down the formula for a general rescaling, say by a in the horizontal direction and by c in the vertical direction?
5. Can you combine these two transformations? What is the formula for first rescaling by a in the horizontal direction and c in the vertical direction, and then translating by e to the right and by f up?

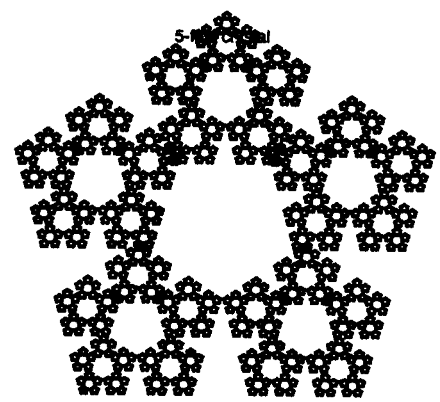
Here are some more pictures of transformations we can do with rescalings and translations and the fractals they generate. See if you can find the formulas for all the transformations below. Then try to make up some transformations of your own.









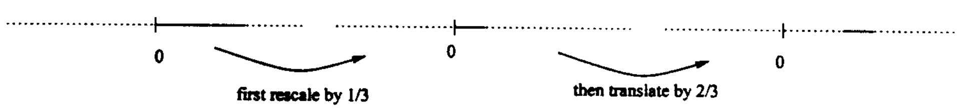


3 Transforming in the plane

Recall that one of the transformations we used to make the Cantor set is

$$T(x) = \frac{1}{3}x + \frac{2}{3}.$$

The picture below describes how T transforms the line; it first rescales by a factor of $1/3$ and then translates to the right by $2/3$.



We have also seen that we can write a combination of translations and rescalings in the plane in the form

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = x \begin{bmatrix} a \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ c \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

This transformation rescales by a factor of a in the horizontal direction and by c in the vertical direction; it also translates to the right by e and up by f.

Let's look more closely at the transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}.$$

This was the third transformation we used to make Sierpinski's triangle. The image of the x-axis under T is the set of points

$$x\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}.$$

This is another horizontal line. The transformation from the x-axis to this horizontal line first rescales by a factor of 1/2 and then translates to the right by 1/4 and up by 1/2. Similarly, the image of the y-axis under T is the set of points

$$y\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}.$$

This is another vertical line. The transformation from the y-axis to this vertical line first rescales by a factor of 1/2 and then translates to the right by 1/4 and up by 1/2.

It might be useful to answer each of the questions below first for the particular transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 2/5 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2/5 \end{bmatrix} + \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix},$$

but answer also the questions for the general transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} a \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ c \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}.$$

1. Show that the transformation sends the x-axis to some line.

2. In which direction does this line point?

3. How does the transformation change the line? For instance, does the transformation rescale the line, changing distances?

4. Answer these same questions for the y-axis.

You should have found that the image of the x-axis is another horizontal line. For the particular example of particular transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\begin{bmatrix} 2/5 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 2/5 \end{bmatrix} + \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix},$$

the x-axis is rescaled by a factor of 2/5, and then translated to the right by 3/5 and up by 2/5. For a more general transformation of the form

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = x\begin{bmatrix} a \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ c \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix},$$

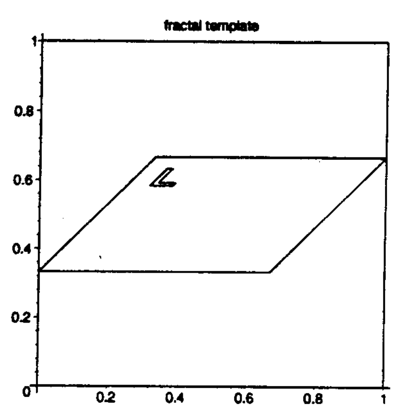
the x-axis is rescaled by a factor of a, and then translated to the right by e and up by f. Similarly, the image of the y-axis is another vertical line.

This is all fine if we only want horizontal and vertical lines, but can we get other lines? Consider the transformation

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = x\begin{bmatrix} 2/3 \\ 0 \end{bmatrix} + y\begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}.$$

1. What is the image of the x-axis under T?
2. What is the image of the y-axis under T?
3. What is the image of the unit square □ under T? Where do the corners of the unit square go?
4. Draw a picture of the image of the unit square under T. Be sure to label the corners.

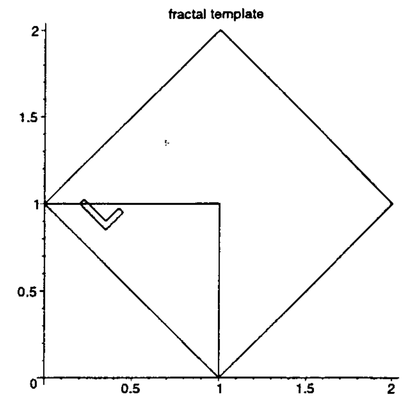
You should have found that the image of the unit square is the following parallelogram:



Now consider a transformation of the form

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = x\begin{bmatrix} a \\ b \end{bmatrix} + y\begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

such that the image of the unit square is the following parallelogram:



Suppose we also know that

$$T\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}\quad T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}\quad T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\2\end{bmatrix}$$

Can you recover the transformation T? (In other words, can you find the coefficients a, b, c, d, e, f?) (Hint: you might want to translate this parallelogram so that $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right)$ lies at the origin.)

You should have found that the transformation is

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=x\begin{bmatrix}1\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}.$$

1. If you're given a transformation of the form

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=x\begin{bmatrix}a\\b\end{bmatrix}+y\begin{bmatrix}c\\d\end{bmatrix}+\begin{bmatrix}e\\f\end{bmatrix}$$

   can you describe the image of the unit square in terms of the coefficients a, b, c, d, e, f? (Hint: start with the corners.)

2. If you're given a parallelogram (with the corners labeled) which is the image of □ under some transformation T, can you find an algorithm to determine T?

3. If you're just given the parallelogram, but the corners are not labeled, can you uniquely determine T?

You Should have found that the image of □ under a transformation of the form

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=x\begin{bmatrix}a\\b\end{bmatrix}+y\begin{bmatrix}c\\d\end{bmatrix}+\begin{bmatrix}e\\f\end{bmatrix}$$

is a parallelogram with corners

$$\begin{bmatrix} e \\ f \end{bmatrix} \quad \begin{bmatrix} a+e \\ b+f \end{bmatrix} \quad \begin{bmatrix} a+c+e \\ b+d+f \end{bmatrix} \quad \begin{bmatrix} c+e \\ d+f \end{bmatrix}.$$

Also, if you're given a parallelogram with corners A, B, C, D (listed in counter-clockwise order) then the transformation T is

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = x(B-A) + y(D-A) + A.$$

Transformations of the form

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = x\begin{bmatrix} a \\ b \end{bmatrix} + y\begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

are called **affine** transformations. They are natural generalizations of transformations of the form $T(x) = mx + b$ of the line.

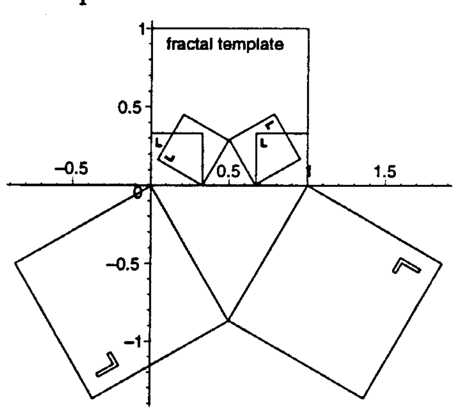
4 Fractal templates using affine maps

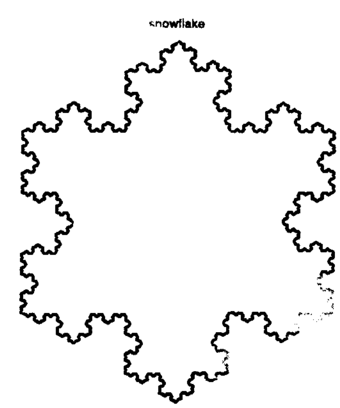
Now we know how to make fractals using affine maps. First of all, we call a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ a **contraction** if it is distance decreasing; i.e. if for $P, Q \in \mathbb{R}^2$ we have the distance between $T(P)$ and $T(Q)$ is less than or equal to k times the distance between P and Q, where $k < 1$. All we have to do is find a finite number of affine maps $T_1, \ldots, T_k$ which are contractions and use the following recipe.

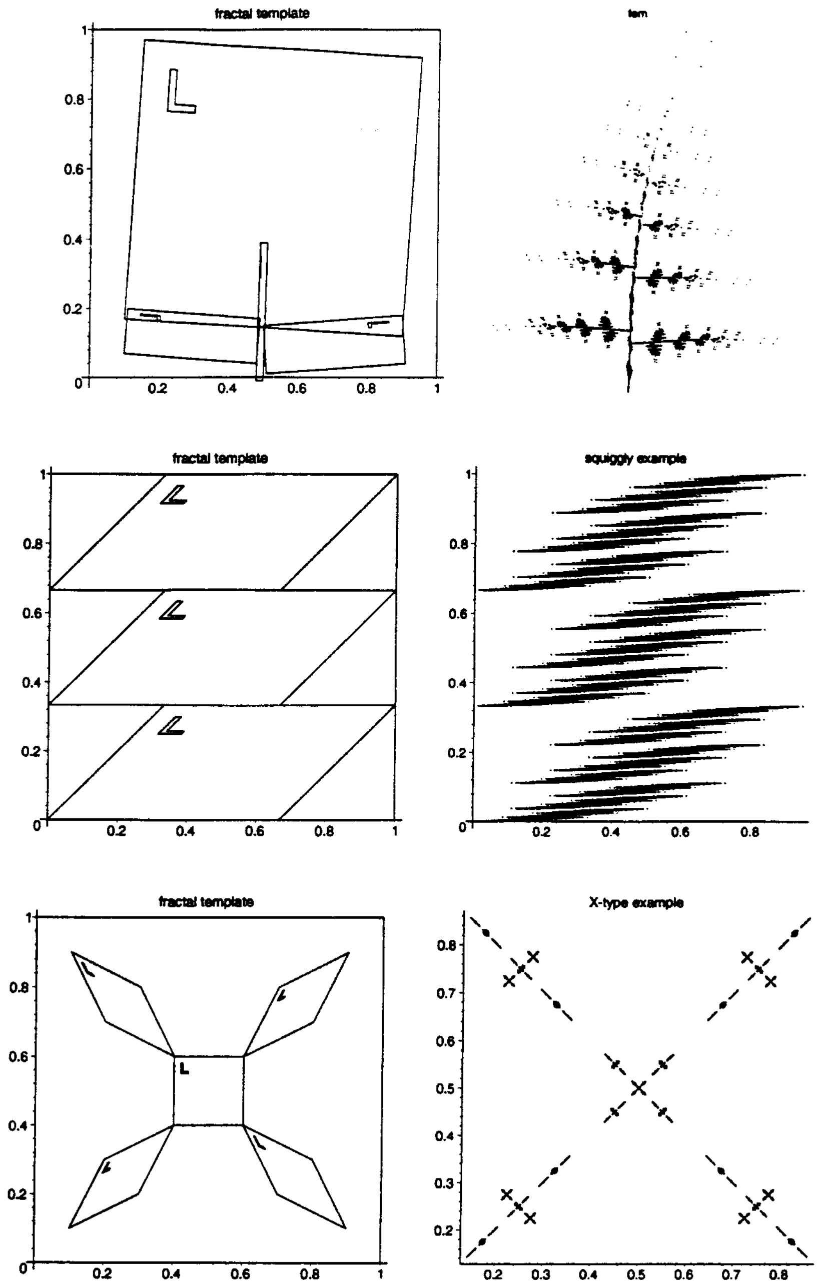
1. Start with any nice bounded set U_0, like a square or even just a point.
2. Replace U_0 with $U_1 = T_1(U_0) \cup \cdots \cup T_k(U_0)$.
3. Repeat.

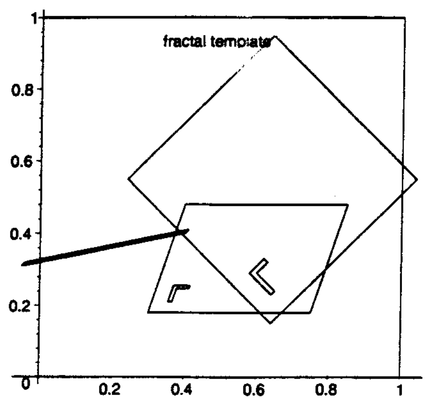
The easiest way to check that all the affine maps are contractions is to force the image of a unit square to be strictly contained in a unit square. (See appendix A for a discussion of why this recipe works.)

Here are some example for you to examine. After you examine these, you can create your own pictures.






fractal template
fern
fractal template
squiggly example
fractal template
X-type example


fractal template
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0.2
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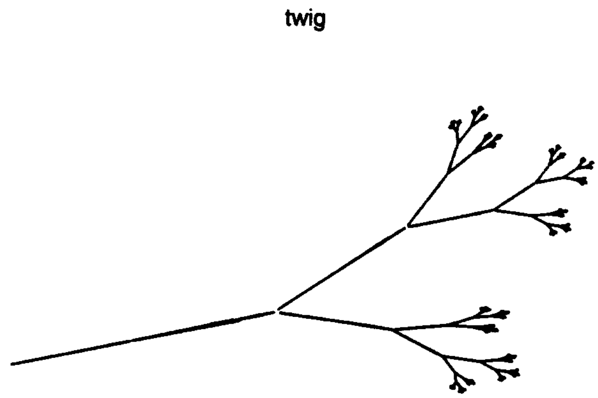
twig

A The contraction mapping principle and another way to make fractals

We will denote the distance between two points P and Q as $||P - Q||$. A **contraction** of the $\mathbb{R}^2$ is any map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $||T(x) - T(x')|| \leq k||x - x'||$ for some $k < 1$. In other words, a contraction is any map which decreases distances by some fixed amount. Suppose T is such a contraction and that $T(0) = 0$.

1. What can you say about $||T(x)||$ as compared to $||x||$?

2. What can you say about the sequence $\{x_n\}$, where $x_{n+1} = T(x_n)$ (i.e. $x_n = T(T(T \cdots (x)))$ for some x)?

You should have found that if T is a contraction such that $T(0) = 0$ then $T(T(T \cdots (x))) \to 0$ as $n \to \infty$, regardless of what x is. The condition $T(0) = 0$ is not a major obstacle. In general, $T(T(T \cdots (x))) \to \bar{x}$ as $n \to \infty$, where $T(\bar{x}) = \bar{x}$. (Why? Look at $||T(x_n) - x_n||$.) The point $\bar{x}$ such that $T(\bar{x}) = \bar{x}$ is called a fixed point of T; all contractions have a unique fixed point, and $T(T(T \cdots (x))) \to \bar{x}$ for any starting point x. This fact about contractions is known as the contraction mapping principle.

We can apply the contraction mapping principle to make some fractals, but first we need to define the **Hausdorff distance** between two sets. Let A and B be two bounded sets in the plane. For $P \in A$, define the distance from P to B by

$$\text{dist}(P, B) = \min_{Q \in B}(||P - Q||).$$

Similarly, for $Q \in B$ we can define the distance between Q and A by

$$\text{dist}(Q, A) = \min_{P \in A}(||P - Q||).$$

Then we can define the Hausdorff distance between the sets A and B by

$$\text{dist}(A, B) = \max[\max_{P \in A}(\text{dist}(P, B)), \max_{Q \in B}(\text{dist}(Q, A))].$$

One can think of the Hausdorff distance as measuring how far you have to move the set A so that it coincides with the set B. A good example to think about is when A and B are two concentric discs with the radius of A being 1 and the radius of B being 2. Then the Hausdorff distance between A and B is 1. A sequences of sets converge to a limit when their Hausdorff distances becomes small. The contraction mapping principle shows that the sequence of bounded sets we generate converge, in the sense that the Hausdorff distance between the iterates becomes very very small as we continue to iterate.

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a contraction. What happens when you apply T over and over again to the unit square?

2. Was there anything special about the unit square? What do you get starting with any bounded shape (e.g. a circle or a rectangle)?

3. Now suppose you have two contractions $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$. This time in the iterative process you replace the unit square U with $T_1(U) \cup T_2(U)$. What happens when you keep doing this?

4. Was there anything special about having two contractions? Does the same idea work with three or four contractions?

Our general recipe for creating fractals is the following. Start with some finite number of contractions $T_1, \ldots, T_k$ and any bounded object (like a square, or a circle, or even a point). Call that bounded object U. Then replace U with $U_1 = T_1(U) \cup \cdots T_k(U)$. Then replace U_1 with $U_2 = T_1(U_1) \cup \cdots \cup T_k(U_1)$. In general, replace U_n with $U_{n+1} = T_1(U_n) \cup \cdots T_k(U_n)$. The limit you obtain is the fractal. The fundamental reason this recipe works is that the iterative step is a contraction on the space of nice, bounded sets in the plane. Then the contraction mapping principle guarantees a fixed point, which is our fractal.