

**NUMBER LOOPS AND PATTERNS**  
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1. GETTING INTO AN ENDLESS LOOP

Try the following algorithm:

- (1) Select a four digit number with not all digits the same.
- (2) Rearrange the digits so that they form the largest number possible.
- (3) Rearrange the digits so that they form the smallest number possible.
- (4) Subtract the smaller number from the larger number. If you end up with a three digit number, put a zero in the thousands place.
- (5) Repeat this process until you notice something quite interesting.

Although it has not been proved (nor disproved), you will always end up with the number 6,174 after a few iterations of this algorithm. Try a few numbers to convince yourself of this unexplainable phenomenon.

Now, we should look at what happens for two, three, and five digit numbers when we perform the same algorithm. After the subtraction, if you end up with a one, two, or four digit number, place a zero in the tens, hundreds, or ten thousands place respectively. Do you notice any patterns or loops occurring? I noticed a few.

- (1) For two digit numbers a loop forms. Furthermore, each number we get after subtraction is divisible by 9 and the sum of digits of the number is equal to 9.
- (2) After a few iterations of the algorithm on three digit numbers, we seem to end up at the number 495 every time. Also, after each subtraction, the sum of the digits of the new number is equal to 18 and the number itself is divisible by 9
- (3) For five digit numbers a loop forms and each new number obtained after subtraction is divisible by 9 and has digits which sum to 27.

You should try to show that my observations are incorrect as they are merely my conjectures and have never been proved to hold true for all two, three, and five digit numbers. You also may want to try to make some conjectures of your own if you notice any patterns which I did not notice. Try some numbers with more digits as well.

An exceptionally interesting thing happens if we change up our algorithm a little bit. And for this one, we do have a way of explaining the phenomenon. Try the following algorithm:

- (1) Choose any three digit number where the units and the hundreds digits are not the same.  
**I will do it with you by selecting the number 347.**
- (2) Reverse the digits of this number you have selected.  
**Reversing the digits of 347 gives us 743.**
- (3) Subtract the larger number from the smaller number.  
**Our calculated difference is  $743 - 347 = 396$ .**
- (4) Once again, reverse the digits of this difference.  
**Reversing the digits of 396 we get 693.**

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<sup>1</sup>Credit needs to be given to Alfred S. Posamentier for his book *Math Wonders to Inspire Teachers and Students*. His book is heavily referenced within this paper.

(5) Now add your last two numbers.

**We then add the last two numbers to get  $396 + 693 = 1089$ .**

What makes this so interesting is that no matter what number you started with, you also got the number 1089 at the end of the algorithm. Why is this? Fortunately, we have an algebraic way of explaining why this works.

Let us represent our arbitrary three digit number  $htu$  as  $100h + 10t + u$ , where  $h$  represents the hundreds digit,  $t$  represents the tens digit and  $u$  represents the units digit. We will assume that  $h > u$ , which is a safe assumption because it is true either in the number you chose or the reverse of it. Then in the subtraction we will have that  $u - h < 0$ . We do not want this to happen, so we will borrow from the tens place in the larger number. This will give us  $10 + u$  in the units place.

Since the tens digits of the two numbers to be subtracted are equal, and 1 was taken away from the tens digit of the larger number, then the value of this digit is now  $10(t-1)$ . Thus we will need to borrow from the hundreds place, leaving  $h-1$  for the hundreds digit of the larger number, in order to enable subtraction in the tens place. This will make the value of the tens digit be  $10(t-1) + 100 = 10(t+9)$  for the larger number. Now we can do our subtraction.

$$\begin{array}{r} 100(h-1) \quad +10(t+9) \quad +(u+10) \\ \underline{100u \quad +10t \quad +h} \\ 100(h-u-1) \quad +10(9) \quad +u-h+10 \end{array}$$

Reversing the digits of this difference gives us

$$100(u-h+10) + 10(9) + (h-u-1).$$

Now, if we add these two numbers, we get

$$100(9) + 2 \times 10(9) + 9 = 1,089.$$

The number 1,089 is a pretty amazing number. You should experiment with it a bit to see if you can find any number patterns. For example, multiply it by the numbers 1 through 9 and see what sort of patterns emerge from the products.

## 2. SOME COOL DIVISORS

Now, I would like to go over some divisor properties of numbers because we saw hints of these in the last section. These can be very helpful when you are not allowed to use a calculator but you have to reduce a large fraction. Some of you may know these already.

- (1) If the sum of the digits of a number is divisible by 3, then the number itself is divisible by 3. Prove this.
- (2) If the sum of the digits of a number is divisible by 9, then the number itself is divisible by 9. Prove this.

It is interesting to notice that if a number is divisible by 3 or 9 and it is an even number, then it is also divisible by 6 and 18 respectively.

The next cool divisor property needs a bit of development. If you multiply the numbers 7, 11, and 13 together you will get 1,001. If you take any three digit number and multiply it by 1,001 you will get a six digit number composed of two repeating sequences of three digit numbers. For example, if the number is 634, then  $634 \times 1,001 = 634,634$ . What does this imply about the divisors of a six digit number composed of two repeating sequences of three digit numbers? It implies that any number of this form is divisible by 7, 11, and 13.

What happens if the three digit number we choose to multiply by 1,001 has repeating digits? We will get a six digit number with repeating digits. What can be said of its divisors? Are they the same

as above? Are there any more we can *always* add to the list of divisors above? Try a few six digit numbers with repeating digits.

### 3. POWER AND FACTORIAL LOOPS

Try this algorithm:

- (1) Choose any number.
- (2) Cube each of its digits.
- (3) Add the cubed digits.

There are five numbers for which we will get back our original number with just one iteration of this algorithm. These are 1, 153, 370, 371, and 407. For all other numbers, we will fall into a number loop. Sometimes our loop will give us back the original number and sometimes the loop will not contain our number at all. Try a few to see this.

Now, we should examine what happens for the sum of the squares of the digits of a number. Do any loops form? It turns out that we will get loops and what is even more interesting is that we will always end up getting the number 89 in our loop. As a matter of a fact, once we reach the number 89, we will know that our loop has begun. Try a few numbers to show yourself this works.

Recall the definition of the factorial function. For all whole numbers  $n = 0, 1, \dots$ , we define  $n! = n(n-1)(n-2) \cdots 3(2)(1)$ . It is accepted amongst mathematicians that  $0! = 1$  by definition.

Now choose a number, find the factorial of each of its digits, and then add them. You will get back your original number after one cycle, two cycles, or three cycles for certain numbers. Otherwise, a loop will begin.

- (1) One cycle numbers: 1, 2, 145, 40585
- (2) Two cycle numbers: 871, 872, 45361, 45362
- (3) Three cycle numbers: 169, 363601, 1454

It turns out that there are no other such numbers less than 2,000,000 for which we will get back our original number after a certain number of cycles. Try a few numbers not on the cycles list to see if any sort of loop or pattern emerges. All of the numbers I tried ended up coming around to a loop that starts with 169. See if the same thing happens for you.

### 4. NUMBER PATTERNS

What happens when we multiply numbers with all ones for digits?

$$\begin{array}{rcl}
 1 \cdot 1 & = & 1 \\
 11 \cdot 11 & = & 121 \\
 111 \cdot 111 & = & 12,321 \\
 1,111 \cdot 1,111 & = & 1,234,321 \\
 11,111 \cdot 11,111 & = & 123,454,321 \\
 111,111 \cdot 111,111 & = & 12,345,654,321 \\
 1,111,111 \cdot 1,111,111 & = & 1,234,567,654,321 \\
 11,111,111 \cdot 11,111,111 & = & 123,456,787,654,321 \\
 111,111,111 \cdot 111,111,111 & = & 12,345,678,987,654,321
 \end{array}$$

Now the question is how does this happen? It has something to do with the fact that our number system is a base 10 system. Try working out the algebra behind it and find an algorithm for the development of our pattern.

Here is another beautiful pattern that involves numbers with all ones for digits and the number 9.

$$\begin{array}{rcl}
 0 \cdot 9 & +1 = & 1 \\
 1 \cdot 9 & +2 = & 11 \\
 12 \cdot 9 & +3 = & 111 \\
 123 \cdot 9 & +4 = & 1,111 \\
 1,234 \cdot 9 & +5 = & 11,111 \\
 12,345 \cdot 9 & +6 = & 111,111 \\
 123,456 \cdot 9 & +7 = & 1,111,111 \\
 1,234,567 \cdot 9 & +8 = & 11,111,111 \\
 12,345,678 \cdot 9 & +9 = & 111,111,111 \\
 123,456,789 \cdot 9 & +10 = & 1,111,111,111
 \end{array}$$

Again, we want to know how this happens. What is it about our number system that makes this pattern emerge? It may or may not be clear that it has to do with multiples of the number 9. Work on the algebra behind this and try to determine an algorithm for the development of this pattern.

Another version of this gives us very different results. Check out the following:

$$\begin{array}{rcl}
 0 \cdot 9 & +8 = & 8 \\
 9 \cdot 9 & +7 = & 88 \\
 98 \cdot 9 & +6 = & 888 \\
 987 \cdot 9 & +5 = & 8,888 \\
 9876 \cdot 9 & +4 = & 88,888 \\
 98765 \cdot 9 & +3 = & 888,888 \\
 987654 \cdot 9 & +2 = & 8,888,888 \\
 9876543 \cdot 9 & +1 = & 88,888,888 \\
 98765432 \cdot 9 & +0 = & 888,888,888 \\
 987654321 \cdot 9 & -1 = & 8,888,888,888
 \end{array}$$

For those of you who are curious, you should explore what makes this pattern happen. For now though, I would like to show you what happens when we multiply numbers with all ones for digits with numbers with all eights for digits.

$$\begin{array}{rcl}
 1 \cdot 8 & = & 8 \\
 11 \cdot 88 & = & 968 \\
 111 \cdot 888 & = & 98,568 \\
 1,111 \cdot 8,888 & = & 9,874,568 \\
 11,111 \cdot 88,888 & = & 987,634,568 \\
 111,111 \cdot 888,888 & = & 98,765,234,568 \\
 1,111,111 \cdot 8,888,888 & = & 9,876,541,234,568 \\
 11,111,111 \cdot 88,888,888 & = & 987,654,301,234,568 \\
 111,111,111 \cdot 888,888,888 & = & 987,654,321,791,234,568
 \end{array}$$

Now, you should, once again, explore the algebra behind this amazing pattern.

I will leave you with two more cool patterns. But you have to figure them out. What happens when you multiply the number 76,923 by the digits 1 through 12? And, finally, what happens when you multiply the number 14,857 by the numbers 1 through 8? Enjoy hunting for number patterns!