COUNTING INFINITE SETS

Peter Trapa

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There are three kinds of people in the world:
those who can count and those who can't.

Everyone knows what infinity is: it's something that goes on and on forever. But it's often necessary to make a more precise definition. That is the first goal of these notes.

By way of motivation, consider the famous infinite hotel. It has rooms numbered 1, 2, 3, and so on; but there is no largest room number. Even when all the rooms are occupied, the proprietor still displays the "Vacancy" sign. The reason? If a new guest arrives, the proprietor simply tells all the current occupants to move to the next higher room number. More precisely the proprietor tells the occupant of room n to move to room $n+1$. After this is done for all n, each existing guest has his own room, and room number 1 is vacant for the new guest to occupy.

This trick could never work with a finite hotel — when all the room are filled, there is no way to accommodate a new guest (without putting two guests in the same room). We are going to define the notion of infinity so that the converse also holds: this trick will *always* work with an infinite hotel. First we need a few preliminary definitions.

Definition Suppose S and T are sets and that $f : S \to T$ is a function. We say that f is one-to-one if f never sends two points of S to the same point in T; that is,

$$f \text{ is } \textit{one-to-one} \text{ if whenever } f(a) = f(b), \text{ then } a = b.$$

We say that f is onto if every point of T is hit by f; that is

$$f \text{ is } \textit{onto} \text{ if for all } t \in T, \text{ there exists } s \in S \text{ such that } f(s) = t.$$

If $f : S \to T$ is one-to-one and onto, we say that f puts S and T in *one-to-one correspondence.*

A function which is one-to-one is sometimes called *injective*; one that is onto is often called *surjective*; and one that is one-to-one and onto is called *bijective.* This is simply a matter of terminology.

Exercises

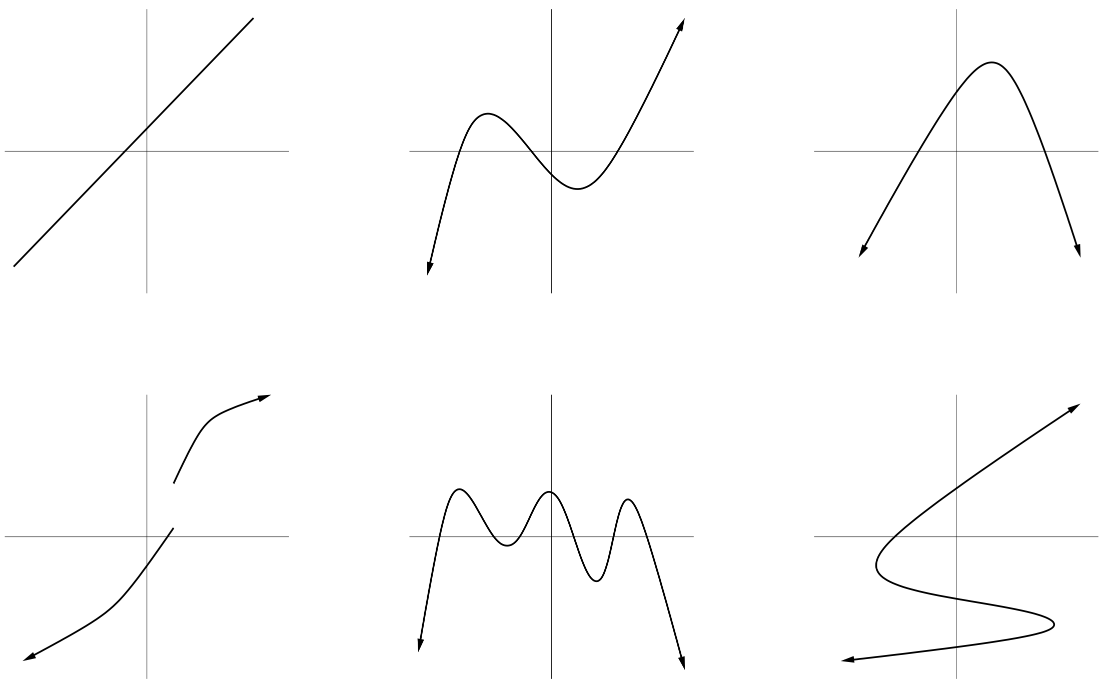
1. Write $\mathbb{N}$ for the set of natural numbers $\{1, 2, 3, \dots\}$. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$. Is f onto? Is f one-to-one?

2. Write $\mathbb{Z}$ for the set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n + 1$. Is f onto? Is f one-to-one?

3. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$. Is f onto? Is f one-to-one?

4. Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n^2$. Is f onto? Is f one-to-one?

5. The following represent graphs of functions from the real numbers $\mathbb{R}$ to $\mathbb{R}$. Decide which are one-to-one, which are onto, which are neither, and which are both.



6. Find a one-to-one onto map

$$\{0, 1, 2, 3, \dots\} \longrightarrow \{1, 2, 3, \dots\}$$

7. Find a one-to-one onto map from the real numbers x such that $x \geq 0$ to the set of real number x such that $x > 0$.

8. Find a one-to-one onto map from the real numbers x such that $0 \leq x \leq 1$ to the set of real number x such that $0 < x < 1$.

Let's return to the notion of infinity. We can now make a precise definition.

Definition. A set S is called *infinite* if there is a map $f : S \to S$ such that f is one-to-one but *not* onto. Here is another way to say the same thing. A set S is called infinite if and only if it there is a subset $T \subset S$ with $T \neq S$ and a one-to-one map $f : S \to T$.

This definition captures our intuitive notion of what it means to be infinite. For example look as the set of rooms in the infinite hotel $S = \{1, 2, 3, \dots\}$. Define a map $f : S \to S$ by $f(j) = j + 1$. (This is the map that the proprietor used.) This is clearly one-to-one: if

$f(j) = j(k)$, then $j+1 = k+1$ and $j = k$. But it's not onto since there is no j such that $f(j) = 1$. So the set of rooms in the infinite hotel is indeed infinite!

The next issue we want to address is the notion of the "size" of a set. First suppose S and T are finite sets. Then S and T have the same number of elements if and only if there is a one-to-one and onto map between them. (Stop and make sure that you really understand this assertion.) So, in the case of finite sets, we say that S and T have the same size if and only if there is a bijection between S and T. Now we may simply extend the definition to arbitrary sets: two sets S and T have the same size if there is a one-to-one onto map between them. (As a matter of terminology the technical word that is often used for "size" is "cardinality." For example, we say that two sets have the same cardinality if there is a bijection between them.)

It may surprise you that there are different sizes of infinite sets. It's convenient to introduce a little more terminology at this point. Let's write $\mathbb{Z}$ for the set $\{\dots,-2,-1,0,1,2\dots\}$. We say that a set S is *countable* if there exists an onto map $f : \mathbb{N} \to S$. For example, if S is finite, we can simply label its elements $\{s_1, s_2, \dots, s_N\}$ and then the function f can be defined as

$$f(j) = \begin{cases} s_j & \text{if } 1 \leq j \leq N \\ s_1 & \text{otherwise.} \end{cases}$$

So finite sets are countable. Of course $\mathbb{N}$ is countable too. To test your understanding, it's a good exercise to verify that $\mathbb{Z}$ is also countable.

Are there other infinite sets that are uncountable? Here is a beautiful trick (called Cantor's diagonal argument) to show that the set $\mathbb{R}$ of real numbers is uncountable. In fact we will show that the interval of real numbers between 0 and 1 is uncountable. Suppose f is any map from $\mathbb{Z}$ to $[0,1]$. Our task is to show that f cannot be onto. Then we will have proved $[0,1]$ (and hence $\mathbb{R}$) is uncountable. Consider the value $f(1)$. This is a real number, so we can express it in decimal notation and write

$$f(1) = .x_1^{(1)} x_2^{(1)} x_3^{(1)} \cdots ;$$

here each $x_j^{(i)}$ is just a number between 0 and 9. Let's list the other values of f in this way

$$\begin{aligned} f(1) &= .x_1^{(1)} x_2^{(1)} x_3^{(1)} x_4^{(1)} x_5^{(1)} \cdots \\ f(2) &= .x_1^{(2)} x_2^{(2)} x_3^{(2)} x_4^{(2)} x_5^{(2)} \cdots \\ f(3) &= .x_1^{(3)} x_2^{(3)} x_3^{(3)} x_4^{(3)} x_5^{(3)} \cdots \\ f(4) &= .x_1^{(4)} x_2^{(4)} x_3^{(4)} x_4^{(4)} x_5^{(4)} \cdots \\ f(5) &= .x_1^{(5)} x_2^{(5)} x_3^{(5)} x_4^{(5)} x_5^{(5)} \cdots \\ &\vdots \end{aligned}$$

Now choose numbers y_j from 0 to 9 so that each y_j differs from the diagonal element $x_J^{(j)}$,

$$y_j \neq x_j^{(j)} \text{ for all } j.$$

Consider

$$y = .y_1 y_2 y_3 y_4 y_5 \cdots .$$

Clearly $y \in [0,1]$. But by construction there is no integer k such that $f(k) = y$. So f cannot be onto. So $[0,1]$ is uncosuntable! Thus the interval $[0,1]$ does not have the same size as $\mathbb{Z}$!

Here are some problems to test your understanding of countability.

Exercises

1. Is the set of pairs of integers countable?

2. Is the set of rational numbers $\mathbb{Q}$ (i.e. fractions) countable?

3. Is the set of irrational numbers countable?

4. Is the set of real numbers x such that $0 \leq x \leq 1$ countable?

Aside. One of the great problems of the last hundred years is called the continuum hypothesis. It can be stated as follows.

Conjecture. *Any set of real numbers is either countable or can be put in one-to-one correspondence with the entire set of real numbers.*

Here are two harder examples that we will discuss in the course of today's Circle.

Question. Does the interval $(-1, 1)$ have the same size as the entire real line?

Question. Does the set of point lying in a square of edge-length one have the same size as the interval $[0, 1]$?

As a final example, we consider the Cantor set. We start with the interval of real numbers from 0 to 1 and remove the middle interval from 1/3 to 2/3,

$S_1 =$

Then perform the same procedure to each of the remaining intervals to arrive

$S_2 =$

Continue in this way,

$S_3 =$

$S_4 =$

$S_5 = ...\ \ . . . . \qquad . . . .\ \ . . . . \qquad\qquad . . . .\ \ . . . . \qquad . . . .\ \ . . . .$

Finally define

$$S = \bigcap_i S_i.$$

This is called the Cantor set. It looks like a little dust on the real line, and doesn't look very infinite at all. Formally we can ask:

Question. Is S countable?