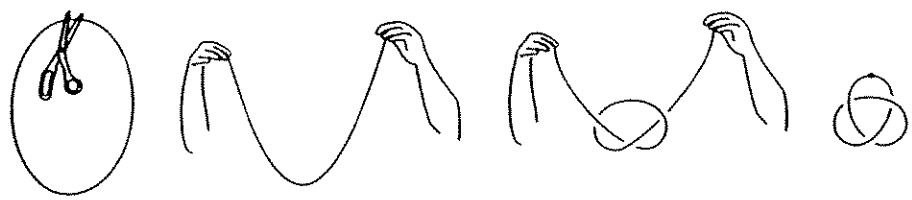
KEY

1. Still Knot A Problem

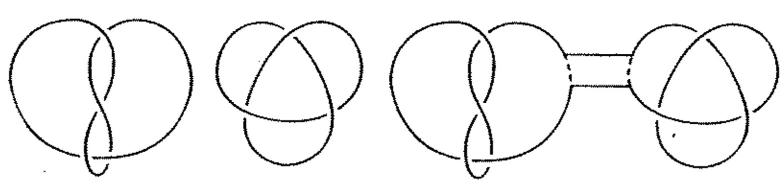
Let's review our "knot theory" from last week. Tie a knot in a piece of string. Glue the two ends of the string together. The result is a string that has no loose ends and is truly knotted. Unless we use scissors, there is no way we can untangle the string. This knotted loop of string is what mathematicians call a **knot**. This material is taken from a great book called The Knot Book [?] .



We think of the knot as being made of flexible rubber. If we move the knot around, stretch it, twist it, or bend it, we still have *the same* knot. That is, we do not distinguish between the original knot and a deformation of it.

Definition 1. *Projection*
We call a picture of a knot a projection of the knot. We learned last week that a knot has many, many different projections (we take a knot, twist a strand, and get a new projection of the same knot).

Last week we saw how to "add" knots together. We saw how knots behaved somewhat like integers, with *prime* knots and *prime* integers. They differ from integers in that *for some* knots, it matters *where* the cut is introduced.



J *K* *J#K*

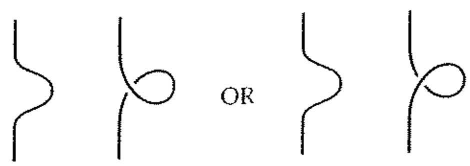
We spent a lot of time examining the *moves* of a knot which simplify (or complicate) its projection while leaving the knot unchanged.

2. Reidemeister Moves

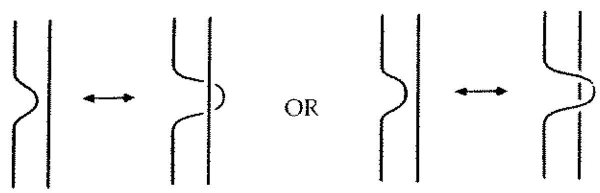
Given two projections of the same knot, we should be able to rearrange the strings so that the projections are the same. This rearranging would take place in our 3-dimensional world and would not allow for one strand to pass through another strand.

Definition 2. *Reidemeister move*
One of three ways to change a projection of a knot that ***will*** *change the relation between the crossings. Any two projections of the same knot can be obtained from the other by a series of Reidemeister moves.*

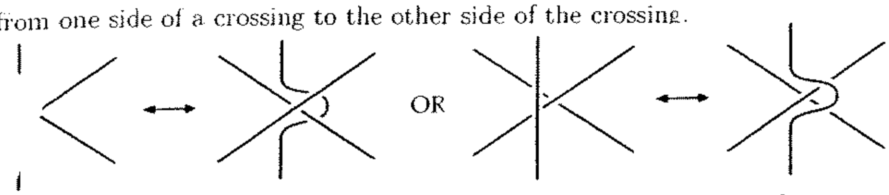
2.1. Type I Reidemeister move.
The first move allows us to put in or take out a twist in the knot.



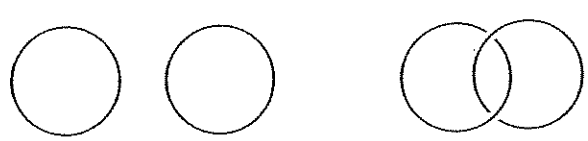
2.2. Type II Reidemeister move.
The second move allows us to either add two crossings or move two crossings.



2.3. Type III Reidemeister move.
The third move allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing.



The ideas we have seen are not limited to just single knots. For example, we can consider two or more knots which are *linked* together.



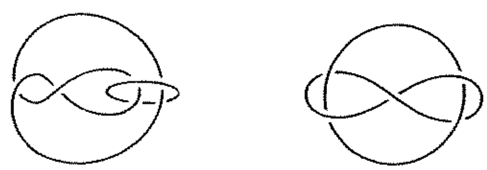
The unlink of two components and the Hopf link.

3. Links

A **link** is a set of knotted loops all tangled up together. Two links are considered the same if we can deform one link to the other without any strand-intersections.

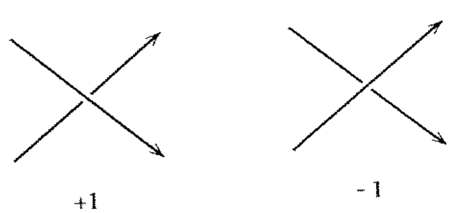
3.1. **Exercise.** The Borremean Rings (or link) consist of three mutually interlocked loops (rings) with the property that removing any one of the rings makes the link fall apart. That is, no two rings are linked. Sketch a projection of the **Borremean Rings**.

3.2. **Exercise.** Do the two projections below represent the same link?



3.3. **Link Invariants.** We want to describe some feature of a link which is independent of the projection.

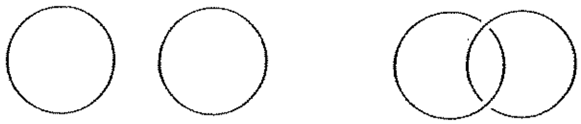
One way to measure numerically "how linked up" components of a link are is by **linking number**. We compute the linking number of a link at each intersection *of two components* by the rule shown in the figure. Then, add up the resulting +1's and -1's and divide by two. This is the linking number.



Computing linking number.

I WROTE THIS PAGE OF SOLUTIONS LAST WEEK — CHECK SOLUTIONS THERE

3.4. **Exercise.** Compute the linking number of the unlink, the Hopf link, and the Borremean link.

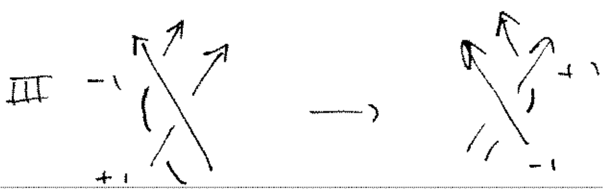


(LAST WEEK'S PAGE)

3.5. **Exercise.** Show the linking number is an invariant of an oriented link. That is, once the orientations are chosen on the two components of the link, the linking number is unchanged by deformations of the link.

Type I moves are done away from link crossings.

⇒ L(II) = 0



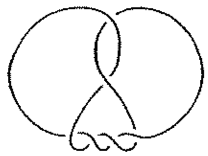
NO CHANGE. BOTH ARE ZERO.

4. Tri-colorability

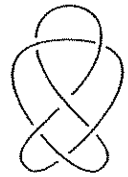
A **strand** in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between.

Definition 3. *Tricolorable*
A projection of a knot or link is tricolorable if each of the strands in the projection can be colored one of three different colors , so that at each crossing, either three different colors come together or all the same color comes together. We also require that at least two colors are used.

4.1. **Exercise.** Which of these knots are tricolorable?



6_1



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(LAST WEEK)

4.2. **Exercise.** Show that the Reidemeister moves preserve tricolorability.

(LAST WEEK)

By your hard work in the above exercise, you have shown that either every projection of a knot is tricolorable or no projection of that knot is tricolorable. For example, every projection of the trefoil knot is tricolorable, and every projection of the unknot is not tricolorable. Can we use tricolorability to conclude that the figure-8 knot is is not the unknot?

5. Knots and Polynomials

We are still seeking a good knot invariant which can tell us whether two knots given to us are the same. We find such an invariant in *knot polynomials*. We will compute the polynomial directly from a projection of the knot, but any two different projections of the same knot will yield the same polynomial. So the **polynomial** is an **invariant** of the knot.

5.1. Bracket Polynomial. $< K >$ denotes the bracket polynomial of a knot K. We create this polynomial using a few (common sense) rules.

Rule 1: $<\bigcirc> = 1$

Next we want a way for determining the bracket polynomial of a knot in terms of the bracket polynomial of simpler knots. We make the bracket polynomial of our original knot a *linear combination* of the bracket polynomials of our new, more simple knots. We do not yet know what coefficients to use, so let's call them A and B.

Rule 2: $<\times> = A <)(> + B <\asymp>$

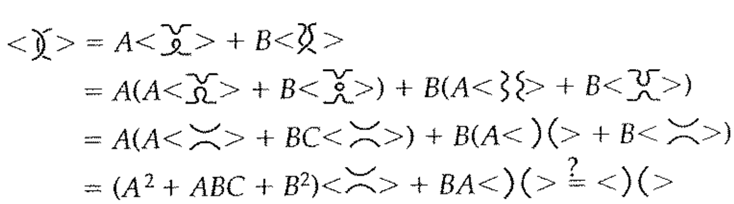
$<\times> = A <\asymp> + B <)(>$

Finally, we want a rule which adding in a trivial component to a link.

Rule 3: $<L \cup \bigcirc> = C <L>$

For our polynomial to be useful, it must be unchanged by the Reidemeister moves.

5.2. Exercise. Show that the bracket polynomial is unchanged by a Type II Reidemeister move. (This will put some restriction on our coefficients.)



Effect on bracket polynomial of Type II move.

The result of this exercise gives us a reformulation of our rules.

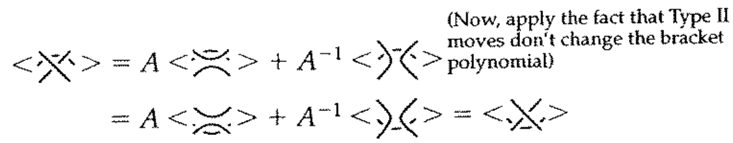
Rule 1: $<\bigcirc> = 1$

Rule 2: $<\times> = A <)(> + A^{-1} <\asymp>$

$<\times> = A <\asymp> + A^{-1} <)(>$

Rule 3: $<L \cup \bigcirc> = (-A^2 - A^{-2})<L>$

5.3. **Exercise.** Show that the bracket polynomial is unchanged by a Type III Reidemeister move. (Easier than for Type II moves.)



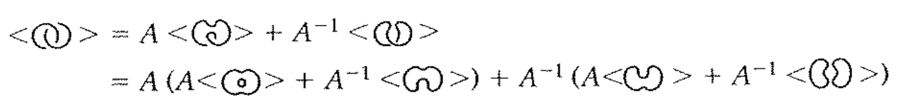
Effect on bracket polynomial of Type III move.

5.4. **Before we look at the Type I Reidemeister moves, let's first flex our muscles.**

Exercise. Find the bracket polynomial :

$$<\bigcirc \cup \bigcirc> = -(A^{-2} + A^2)\ <\bigcirc>\ = -(A^2 + A^{-2})1$$

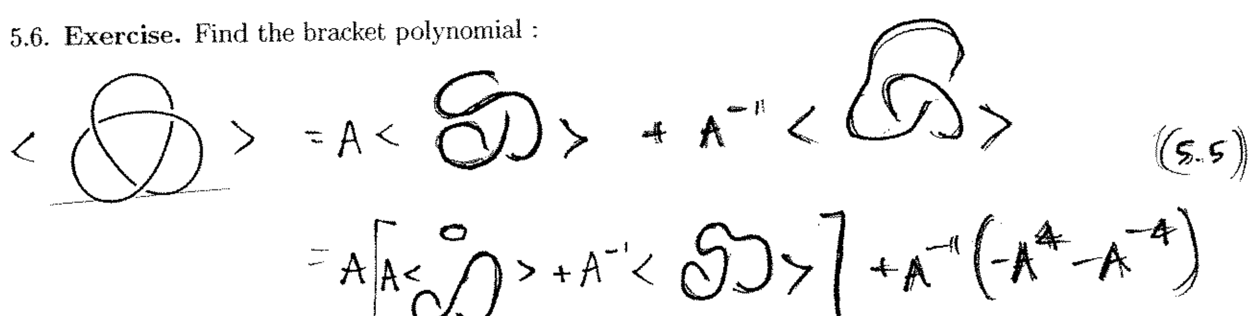
5.5. **Exercise.** Find the bracket polynomial :



$$= A\,(A(-(A^2 + A^{-2})) + A^{-1}(1)) + A^{-1}(A(1) + A^{-1}(-(A^2 + A^{-2})))$$

$$= -A^4 - A^{-4}$$

5.6. **Exercise.** Find the bracket polynomial :

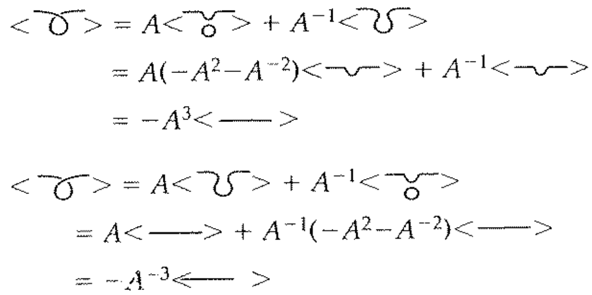


$$= A\left[A(-A^2 - A^{-2})\left[A<\circ\circ> + A^{-1}<\circ>\right] + A^{-1}\left[A<\circ\circ> + A^{-1}<\circ>\right]\right] + (-A^3 - A^{-5})$$

$(A(-A^2 - A^{-2}) + A^{-1}) = -A^3$

$$= -A^7 - A^3 - A^{-5}$$

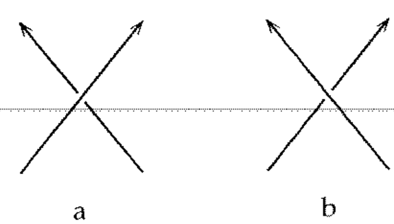
5.7. **Exercise.** Find the bracket polynomials and compare the results :



Effect on bracket polynomial of Type I move.

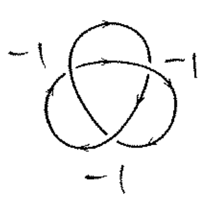
We notice that the two polynomials (which we hoped would be the same) are different just by a sign. If our knots are the same, but twisted in different directions, the bracket polynomial picks up on this. We see that the bracket polynomial is not the perfect invariant we were looking for - but it's close. We need to add one more ingredient to the mix.

Given a knot, we orient it and compute the *twist number* as follows. At each crossing of an oriented knot projection we have either a +1 or -1 as in the figure below. (Note if we rotate the understrand clockwise to match the overstrand we count this as a +1, and a counterclockwise rotation gives a -1.)



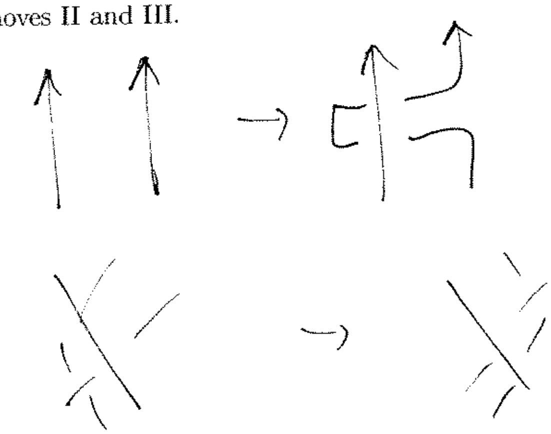
(a) +1 crossing. (b) −1 crossing.

5.8. **Exercise.** Find the twist number.



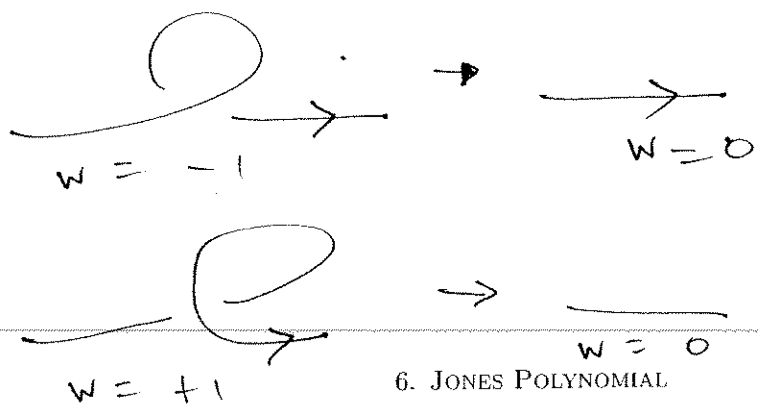
$\omega(T) = -3$

5.9. **Exercise.** Show that the twist of a link projection is invariant under Reidemeister moves II and III.



same as linking number —.

5.10. **Exercise.** What is the effect on the twist number of a Type I Reidemeister move?



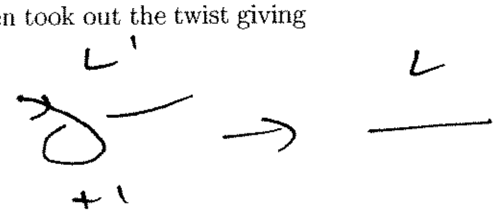
changes by ±1

6. Jones Polynomial

We incorporate the bracket polynomial into a new polynomial which accounts for the twisting of knot.

$$X(L) = (-A^3)^{-w(L)} < L >$$

We've already checked that $w(L)$ and $< L >$ are unaffected by Type II and III moves, so $X(L)$ is unaffected as well! What happens to $X(L)$ after a Type I move? Suppose first we had a strand as in the figure below in a projection L', and we then took out the twist giving us a new projection L.

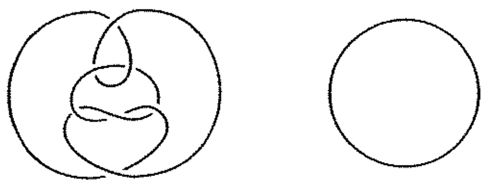


Then $w(L') = w(L) + 1$, so

$$\begin{aligned}
X(L') &= (-A^3)^{-w(L')} < L' > \\
&= (-A^3)^{-(w(L)+1)} < L' > \\
&= (-A^3)^{-(w(L)+1)}((-A^3) < L >) \\
&= (-A^3)^{(-w(L))} < L > \\
&= X(L)
\end{aligned}$$

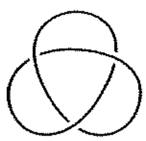
Hooray! The new Jones polynomial $X(L)$ is unaffected by this Type I Reidemeister move. (It is also unaffected by the other version of a Type I move.) **So, $X(L)$ is an invariant for knots and links.**

6.1. **Example.** Last week we found Reidemeister moves which transformed the following projection into the unknot. By all of our hard work, we know that the Jones polynomial of both projections is simply 1. (Try it, if you don't believe me.)

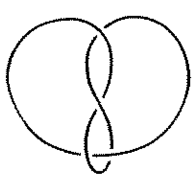


Twist #

6.2. **Exercise.** Find the Jones polynomial $X(L)$. of the two knots shown below. — BRACKET

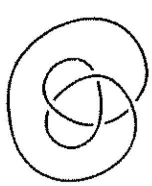
X() $= -A^{3^{(-3)}}\left(-A^{3} - A^{-5} - A^{7}\right)$

$= A^{12} + A^{14} + A^{16}$



similar to trefoil.

First find Bracket Polynomial. Then Twist #.

X() = X() (Invariant under Reidemeister Moves)

$= A^{12} + A^{14} + A^{16}$

References

[Adams] C. Adams. The Knot Book *An Elementary Introduction to the Mathematical Theory of Knots W.H. Freeman and Company*(1994).