MATH CIRCLE CONTEST SOLUTIONS

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1. SECRET ADMIRERS?

```
            B
           E E
          M M M
         Y Y Y Y
        V V V V V
       A A A A A A
      L L L L L L L
       E E E E E E
        N N N N N
         T T T T
          I I I
           N N
            E
```

In the above diagram, by beginning at the top "B" and moving down, one row at a time, to adjacent letters, how many ways can one spell BEMYVALENTINE? (For instance, one possibility would be to begin at the "B" move down and left for 6 rows to spell "BEMYVAL" and then move down and right for 6 more rows to finally spell "BEMYVALENTINE".)

Solution. The problem asks for the number of (appropriately defined) paths from B in E in the above diagram. If we start tallying the number of such paths, as easy pattern emerges

```
   1
  1 1
 1 2 1
1 3 3 1
```

So the number we are seeking is really the 7th entry in the 13th row of Pascal's triangle. We can either compute this iteratively or recall the formula,

$$\text{the } i\text{th entry in the } n\text{th row of Pascal's triangle is } \binom{n-1}{i-1}.$$

So the answer is

$$\binom{12}{6} = 924.$$

2. FAIR TOSS

Suppose you play a game wherein you roll five dice, and score the total of the highest three. What is the probability that you score 16 or more?

Solution. This is a bit of a nasty count. First notice that there are four ways to win: `666xx`, `665yy` (with `y`$\neq 6$), `655zz` (with `z` $\neq 6$), and `664ww` (with `w` $\neq 5, 6$). We have to count each of them separately.

`666xx`. We break this up into three sub-counts: the number of tosses with exactly three 6's, the number with exactly four 6's, and the number with all 6's. For exactly three 6's, there are $10 = \binom{5}{3}$ choices for the location of the 6's, and 5^2 possibilities for the remaining numbers (none of which can be a 5). For four 6's, we get $\binom{5}{4}$ possibilities for the location of the 6's and 5 possibilities for the remaining number. Finally there is only one way to throw all 6's. So we get a total of

$$\binom{5}{3} \cdot 5^2 + \binom{5}{4} \cdot 5^1 + \binom{5}{5} \cdot 5^0 = 250 + 25 + 1 + 276$$

possibilities.

`665yy`. As before, we break this up into three sub-counts: the number of tosses with exactly one 5, the number with exactly two 5's, and the number with three 5's.

Arguing as before we get

$$\binom{5}{2} \cdot \binom{3}{1} \cdot 4^2 + \binom{5}{2} \cdot \binom{3}{2} \cdot 4^1 + \binom{5}{2} \cdot \binom{3}{3} \cdot 4^0 = 480 + 120 + 10 = 610.$$

The grand total is

$$480 + 120 + 10 = 610.$$

`655zz`. We can reason as above, breaking the problem into a sum of exactly two 5's, exactly three 5's, and exactly four 5's. We get a total of

$$\binom{5}{1} \cdot \binom{4}{2} \cdot 4^2 + \binom{5}{1} \cdot \binom{4}{3} \cdot 4^1 + \binom{5}{1} \cdot \binom{4}{4} \cdot 4^0 = 480 + 80 + 5 = 565.$$

`664ww`. We can reason as above, breaking the problem into a sum of exactly 1 4, exactly 2 4's, and exactly 3 4's. We get a total of

$$\binom{5}{2} \cdot \binom{3}{1} \cdot 3^2 + \binom{5}{2} \cdot \binom{3}{2} \cdot 3^1 + \binom{5}{2} \cdot \binom{3}{3} \cdot 3^0 = 270 + 90 + 10 = 370.$$

So the final probability is

$$\frac{276 + 610 + 565 + 370}{5^5} = 0.23418.$$

So just under a one-in-four chance.

3. ON AND ON AND ON AND ON...

Determine the value of the following expression:

$$\frac{1}{2+\frac{1}{3+\frac{1}{2+\frac{1}{3+\cdots}}}}.$$

Solution. Let x denote the above expression, and notice that

$$x = \frac{1}{2+\frac{1}{(3+x)}}.$$

Clearing denominators and simplifying, we get

$$2x^2 + 6x - 3 = 0,$$

and so

$$x = \frac{3}{2} \pm \frac{\sqrt{60}}{4},$$

and we take the positive solution

$$\frac{3}{2} + \frac{\sqrt{15}}{2}.$$

If a, b, and c are the roots of $x^3 - 5x^2 + 3x - 2$, find

$$\frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}.$$

Solution. It's difficult to compute the roots a, b, c themselves, but perhaps we can get by with the easy information provided by the coefficients of the cubic equation:

$$a + b + c = 5 \qquad ab + bc + ac = 3 \qquad abc = 2.$$

Let

$$S = \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b}.$$

If we multiply through by abc we get

$$abcS = a^2c + a^2b + b^2c + ab^2 + bc^2 + ac^2,$$

which we can rewrite as

$$\begin{aligned} abcS &= a(ac + ab) + b(bc + ab) + c(bc + ac) \\ &= a(ac + ab) + b(bc + ab) + c(bc + ac) + 3abc - 3abc \\ &= a(ac + ab + bc) + b(bc + ab + ac) + c(bc + ac + ab) - 3abc. \end{aligned}$$

So

$$S = \frac{a(ac + ab + bc) + b(bc + ab + ac) + c(bc + ac + ab) - 3abc}{abc}$$

or

$$S = \frac{(a + b + c)(ab + bc + ac) - 3abc}{abc}.$$

Now we can plug in from the first set of equations listed above to get

$$S = \frac{5 \cdot 3 - 3 \cdot 2}{2} = \frac{9}{2}.$$

5. A FAMILIAR CHALLENGE

Assume

$$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}$$

is an identity in x. Find

$$a_0 + a_2 + a_4 + \cdots + a_{2n}$$

in terms of n.

Solution. Plug in the values $x = 1$ and $x = -1$ to obtain

$$\begin{aligned} 3^n &= a_0 + a_1 + a_2 + \cdots + a_{2n} \\ 1 &= a_0 - a_1 - a_2 + \cdots + a_{2n}. \end{aligned}$$

If we add these equations and divide by 2, we get

$$\frac{3^n - 1}{2} = a_0 + a_2 + a_4 + \cdots + a_{2n}$$