Weighted Graphs

First, we recall the information from last week:

A **directed graph** or **digraph G** consists of a set **V** of **vertices** $\{1, 2, \ldots, n\}$ and a set **E** of edges: pairs of vertices (i, j).

The **incidence matrix** of a digraph is the $n \times n$ matrix M of 1s and 0's where we put a 1 in the (i, j) spot if there is an arrow from j to i; otherwise the entry is zero. That is: $m_{i,j} = 1$ if (i, j) is an edge, $m_{i,j} = 0$ if (i, j) is not an edge.

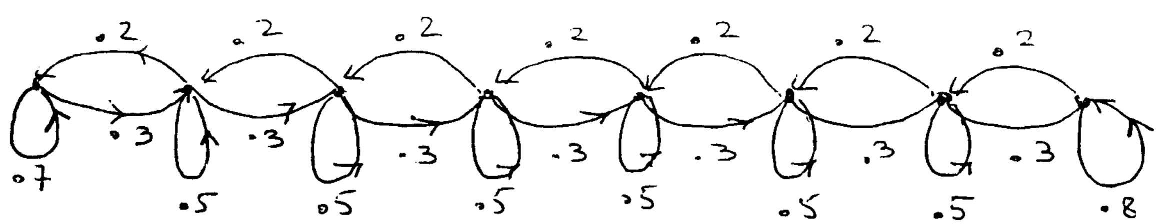
A **path** in a graph is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$. Note that the first term of each edge is the same as the second term of its predecessor. We call k the **length** of the path. A path is called **closed** if its end point is its beginning point: $i_k = i_1$.

The (i, j) entry of M^k (denoted $m^k_{i,j}$) is the number of paths of length k from j to i. For a graph this is the same as the number of paths of length k from i to j. The (i, j) entry of $M + M^2 + \cdots + M^k$ is the number paths of length at most k from i to j. Note that the (i, i) entry of M^k is the number of closed paths of length k starting and ending at i.

Now, we consider the possibility that the paths are weighted.

Suppose that we have a barrel of water 8 feet high, and someone floats red dye on the top so that the first foot of water has 100 grams of dye in it. As time goes by the dye diffuses through the barrel. let's say that, in any hour 30% of the dye diffuses downward from any one-foot layer, 20% diffuses upwards and the remainder stays put. How much dye is in the bottom layer in the tenth hour? In each layer?

Unsurprisingly, these calculations can be transformed into matrix calculations. Number the one-foot layers in the barrel $1, 2, 3, \ldots, 8$, starting at the top. Since, in any hour .5 of the dye stays at any layer, .3 goes down a layer, and .2 goes up, we can represent this by the graph



which in turn is described by the matrix which has, for each i, a .5 in the (i, i) place, a .3 in the $(i, i-1)$ place, and a .2 in the $(i, i+1)$ place (except that there is no upward diffusion in the top layer, and no downward in the bottom:

$$A = \begin{pmatrix} .7 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ .3 & .5 & .2 & 0 & 0 & 0 & 0 & 0 \\ 0 & .3 & .5 & .2 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & .5 & .2 & 0 & 0 & 0 \\ 0 & 0 & 0 & .3 & .5 & .2 & 0 & 0 \\ 0 & 0 & 0 & 0 & .3 & .5 & .2 & 0 \\ 0 & 0 & 0 & 0 & 0 & .3 & .5 & .2 \\ 0 & 0 & 0 & 0 & 0 & 0 & .3 & .8 \end{pmatrix}$$

Now the matrix A shows us how the dye moves in one hour. That is, the entries in the jth column show us how the dye moves from the ith layer to the other layers in one hour. As with the matrices of digraphs, we can continue: the the entries in the jth column of A^2 show us how the dye moves in two hours from the jth layer to the other layers.

It may be worthwhile taking a look at A^{1000}.

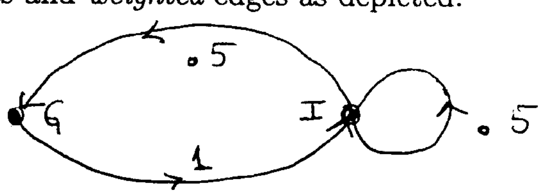
Now, we look some applications for which the associated matrix is not so complex.

Suppose that I am the dictator of an isolated country, which has no trading partners. I gather together all of the industry CEO's and I tell them that I intend to collect from them 1/2 of their revenue in taxes each year, but will transfer to them in services and goods all the state revenue each year.

Is this a stable situation? Let's make a table of the total value of each partner: G (government), I (industry) each year:

YEAR	0	1	2	3	4	5	6	7	
G	0	1/2	1/4	3/8	5/16	11/32	21/64	43/128	...
I	1	1/2	3/4	5/8	11/16	21/32	43/64	85/128	...

Again, we see the rule of matrix multiplication playing a role.We can consider this as a graph with two vertices and *weighted* edges as depicted:



Then this situation is represented by the matrix

$$GI = \begin{pmatrix} 0 & .5 \\ 1 & .5 \end{pmatrix}$$

The first column, as we see, is the distribution of wealth at year zero, the second at year 1. What do you suppose the columns of GI^2 represent:

$$GI^2 = \begin{pmatrix} .5 & .25 \\ .5 & .75 \end{pmatrix}$$

and after 5 years we have

$$GI^2 = \begin{pmatrix} .3125 & .34375 \\ .6875 & .65625 \end{pmatrix}$$

So the sequence of powers GI^k give us the $k-1$ and k columns of this table. What do you suppose GI^{100} looks like?

Answer.

$$GI^{100} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Now, let's look at a more complicated problem: besides government (G) and Industry (I), there is a service industry, S. Again, each year, the government returns to S and I all its worth in equal portion, and collects from each industry half their worth. Finally S and I trade their remaining worth. The matrix corresponding to this is

$$GS = \begin{pmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{pmatrix}$$

We find, after 5 years,

$$GS^5 = \begin{pmatrix} .3125 & .34375 & .34375 \\ .34375 & .3125 & .34375 \\ .34375 & .34375 & .3125 \end{pmatrix}$$

and, after 50 years, all entries are $1/3$.

The (i,j)entry of GS^k shows us how much of the total worth of segment j has been transferred to segment i over the k year period. The fact that eventually all the entries are $1/3$ tells us that eventually each segment ends up with $1/3$ the total wealth, no matter how the distribution began.

These matrices have the following in common:
1) no negative entries
2) the sums of the entries down a column is always 1.

For such a matrix M, it is a fact that its powers M^k stabilize, as k becomes very large, to a matrix all of whose columns are the same. The interpretation is this: the rows and columns represent trading partners in an economy, and the (i,j) entry gives the part of the value of partner j which gets transferred to partner i in one period. Then then the corresponding entry of M^k represents the amount transferred over k periods. The fact that the powers stabilize is the assertion that the economy is stable, with the (common) element in the ith row indicating the portion of the total value held by partner i.

How do we find that column? Well, for a particular matrix M of this type, just multiply it by itself enough times so that the columns become stable. We'll see below a more direct way to calculate this column.

Now, there is a hypothesis missing here: it is essential that there be no subgroup of partners who trade exclusively among themselves. This amounts to saying that the graph whose edges correspond to positive entries is connected.

In the general situation, we suppose that there are n industries, and in each period, the jth industry transfers a portion $p_{i,j}$ of its worth to the ith industry. We then construct the ***transition*** matrix of this economy: the matrix T whose (i,j) entry is $p_{i,j}$. We assume that there is no loss in value, so that the column sums (the jth column tells us how the jth industry distributes its worth) are all 1. Then, so long as there is sufficient mixing, the economy stabilizes; eventually the powers T^k have all columns the same, representing the stable wealth of the economy.

Consider the following matrix, and find (by calculating a high enough power) the stable column.

$$GK = \begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{pmatrix}$$

To see what happens when there is not sufficient mixing, suppose the government is taken over by a dictator who keeps all the taxes for himself. The matrix GS is now replaced by the matrix

$$GD = \begin{pmatrix} 1 & .5 & .5 \\ 0 & 0 & .5 \\ 0 & .5 & 0 \end{pmatrix}$$

Now what happens?

Well, rather than have that happen, S and I decide to pay no more taxes, but to continue trading between themselves. Now we have

$$GE = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- see what happens.

The fact that stability eventually is achieved if there is enough mixing is a famous theorem by Perron, demonstrated in the 1920's and put to use by the Soviets in their attempt to control economy. This theory is now used by ecologists in studying the stability of complex ecosystems. The theorem says this:

Theorem. Suppose that A is an $n \times n$ matrix of non-negative numbers, all of whose column sums are equal to one. Assume "sufficient mixing". Then the matrices A^n converge to a matrix all of whose columns are the same (and have sum equal to 1).

To find this stable column, we argue as follows. Let C be the matrix we are looking for: C has all its columns the same. The defining condition for C is this: $AC = C$, where the product on the left is given by matrix multiplication. For, we know that, if we take a high enough power, $A^{n+1} = A^n = C$; at least to the degree of accuracy of our calculator. If we write this as

$$C = A^{n+1} = AA^n = AC$$

we get the equation $AC = C$. The terminology for this equation is: C is an *eigenvector* for A with *eigenvalue* 1. Let's now see how we can find C from this.

Let

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{pmatrix}$$

be the common column, and let $A = (a_{i,j})$ be the entries of the given matrix. The equation $AC = C$ amounts to the system of linear equations

$$a_{i,1}c_1 + a_{i,2}c_2 + \cdots a_{i,n}c_n = c_i \quad \text{for all } i$$

together with

$$c_1 + c_2 + c_3 + \cdots + c_n = 1 \ :$$

the column sum is 1. Solving this system gives us the answer.

Example: Let

$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$$

and let x, y be the entries of the common column we are looking for. Then we get the equations

$$x + y = 1 \quad \text{(column sum is 1)}$$

$$\frac{1}{2}y = x \ , \quad x + \frac{1}{2}y = 1 \ .$$

From the second and first equations we get

$$\frac{1}{2}y + y = 1\ , \quad \text{so } \frac{3}{2}y = 1\ , \quad \text{so } y = \frac{2}{3}$$

and from that $x = 1/3$.

Try solving the same system for GS, GK, GD.

Challenge Problem.

a) Let a be a positive number less than 1. What is the common column of the limit of the powers of

$$A = \begin{pmatrix} 0 & a \\ 1 & 1-a \end{pmatrix}$$

b) Let a and b be a positive number with $a + b$ less than 1. What is the common column of the limit of the powers of

$$A = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & 1-a-b \end{pmatrix}$$

We can envision this matrix as representing a terrarium with a pond (P); the atmosphere S, and a cactus C. The pond dries up each year, transfering all its moisture to the atmosphere S. The atmosphere in turn, transfers all its moisture to the cactus C. Finally the cactus returns a part a of its moisture to the pond, another part b to the atmosphere, and retains the remainder $1 - a - b$. Is this miniature ecology stable? What is the stable distribution of moisture?

0	1	2	3	4	5	6	7	8	9	10
100	70	55.0	45.7	39.3	34.5	30.7	27.7	25.3	23.2	21.4
0	30	36.0	36.3	34.9	33.1	31.1	29.2	27.5	25.9	24.4
0	0	9.0	15.3	19.1	21.2	22.3	22.7	22.7	22.5	22.1
0	0	0.0	2.7	5.9	8.9	11.2	13.1	14.4	15.4	16.1
0	0	0.0	0.0	0.8	2.2	3.8	5.4	6.9	8.3	9.5
0	0	0.0	0.0	0.0	0.2	0.8	1.5	2.5	3.4	4.4
0	0	0.0	0.0	0.0	0.0	0.1	0.3	0.6	1.1	1.6
0	0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.3	0.5

RED DYE after k HOURS