Solutions for Introduction to Polynomial Calculus

Section 2 Problems - The Slope of a Curve

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(1)

$$\frac{f(1+h)-f(1)}{h} = \frac{3(1+h)+2-(3(1)+2)}{h} = \frac{3h}{h}$$

which equals 3 for $h \neq 0$. The value which any *polynomial* expression in h approaches as h approaches 0 may be determined by setting h equal to 0. Note that before the h is removed from the denominator by finding an expression which is equivalent as long as $h \neq 0$, the expression is *not* a polynomial in h and cannot even be evaluated at $h = 0$.

In this case, the polynomial expression, 3, is a constant and does not even involve h. Evaluating the polynomial $p(h) = 3$ at $h = 0$ gives $p(0) = 3$, so this 'difference quotient' approaches 3 as h approaches 0. Since the curve $y = f(x)$ is a straight line with slope 3, we'd better hope that the slope of a curve computation reduces to the same slope as the line, and indeed it does. Since $f(1) = 5$, The tangent line at $(1, 5)$ is $y - 5 = 3(x - 0)$.

Note on the interpretation and manipulation of expressions of the form $f(x + h)$.: Many students interpret $f(x + h)$ purely symbolically and literally, symbolically replace any occurence of x with $x + h$. This is not a totally unreasonable idea since we teach to 'put what is in the parentheses whereever x is', but is correct in the context. For instance, if $f(x) = 4x$ one might incorrectly write $f(x + h) = 4x + h$, or if $g(x) = x^2$, one might incorrectly write $g(x + h) = x + h^2$. One 'systematic' way to avoid this would be always to replace x by what is between the parentheses *surrounded by parentheses.* In the above examples this would correctly give $f(x+h) = 4(x+h)$ and $g(x+h) = (x+h)^2$. The only problem is for 'simple' arguments in the parentheses it will give strange looking, yet not incorrect, extraneous parentheses, for example $f(a) = 4(a)$ or $g(3) = (3)^2$. You can easily remove these when you are sure they are not needed. An essentially equivalent conceptual approach is to understand the meaning of $f(x) = 4x$ as 'the function which multiplies its input (argument) by 4, so $f(x+h)$ says multiply $x+h$ by 4, and we know 4 times $x+h$ is $4(x + h) = 4x + 4h$ and not $4x + h$. Similarly $g(x) = x^2$ is the function which squares its input, so $g(x+h)$ is the $x+h$ squared, which is $(x+h)^2 = x^2 + 2xh + h^2$, and not $x+h^2$.

The following problems also use the above fact that $(x + h)^2 = x^2 + 2xh + h^2$, and $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. These are special cases of the binomial rule

$$(x + h)^n = \sum_{j=0}^{n} C(n, j)x^{n-j}h^j$$

where $C(n, j)$ is the number of different ways of choosing j objects from a set of n objects when the order does not matter.

See http://www.math.utah.edu/~palais/mst/Pascal.html for a flash application connecting different interpretations of $C(n, j)$ and demonstrating concretely the recursive formula known as Pascal's Triangle, $C(n, j) = C(n - 1, j - 1) + C(n - 1, j)$ and the direct formula for computing $C(n,j) = \frac{n!}{j!(n-j)!}$. (The symbol $n!$, spoken n factorial, represents the product of the positive integers less than or equal to n: $n! = 1 \cdot 2 \cdots n$.

One of the coolest and most powerful results accessible in the first year of calculus is the ability to generalize the binomial rule to the situation where n is not a positive integer, and develop analogous formulas for $\frac{1}{1+x} = (1+x)^{-1}$ and $\sqrt{1+x} = (1+x)^{1/2}$, etc.

(2)

$$\frac{f(0+h)-f(0)}{h} = \frac{h^2-0}{h} = \frac{h^2}{h}$$

which equals h for $h \neq 0$. Evaluating the polynomial $p(h) = h$ at $h = 0$ gives $p(0) = h$, so this 'difference quotient' approaches 0 as h approaches 0. The curve $y = f(x)$ is a parabola with its vertex pointing down at $(0,0)$ and by symmetry, we would expect its slope there would be 0 and indeed it does. The tangent line is horizontal: $y - 0 = 0(x - 0)$.

(3)

$$\frac{f(2+h)-f(2)}{h} = \frac{(2+h)^2-2^2}{h} = \frac{4+4h+h^2-4}{h} = \frac{4h+h^2}{h}$$

which equals $4+h$ for $h \neq 0$. Evaluating the polynomial $p(h) = 4+h$ at $h = 0$ gives $p(0) = 4$, so this 'difference quotient' approaches 4 as h approaches 0. The curve $y = f(x)$ is a parabola. Since $f(2) = 4$, The tangent line at $(2,4)$ is $y - 4 = 4(x-2)$.

(4)

$$\frac{f(1+h)-f(1)}{h} = \frac{(1+h)^2-3-(1^2-3)}{h} = \frac{1+2h+h^2-3-(1-3)}{h} = \frac{2h+h^2}{h}$$

which equals $2+h$ for $h \neq 0$. Evaluating the polynomial $p(h) = 2+h$ at $h = 0$ gives $p(0) = 2$, so this 'difference quotient' approaches 2 as h approaches 0. The curve $y = f(x)$ is a parabola. Since $f(1) = -2$, The tangent line at $(1,-2)$ is $y - (-2) = 2(x-1)$.

(5)

$$\frac{f(0+h)-f(0)}{h} = \frac{h^2+2h-1-(-1)}{h} = \frac{h^2+2h}{h}$$

which equals $h+2$ for $h \neq 0$. Evaluating the polynomial $p(h) = h+2$ at $h = 0$ gives $p(0) = 2$, so this 'difference quotient' approaches 2 as h approaches 0. The curve $y = f(x)$ is a parabola. Since $f(0) = -1$, The tangent line at $(0,-1)$ is $y - (-1) = 2(x-0)$.

(6)

$$\frac{f(1+h)-f(1)}{h} = \frac{3(1+h)^2-2-(3(1)^2-2)}{h} = \frac{3+6h+3h^2-2-(3-2)}{h} = \frac{6h+3h^2}{h}$$

which equals $6+3h$ for $h \neq 0$. Evaluating the polynomial $p(h) = 6+3h$ at $h = 0$ gives $p(0) = 6$, so this 'difference quotient' approaches 6 as h approaches 0. The curve $y = f(x)$ is a parabola. Since $f(1) = 1$, The tangent line at $(1,1)$ is $y - 1 = 6(x-1)$.

(7)

$$\frac{f(1+h)-f(1)}{h} = \frac{(1+h)^3-1^3}{h} = \frac{1+3h+3h^2+h^3-1)}{h} = \frac{3h+3h^2+h^3}{h}$$

which equals $3+3h+h^2$ for $h \neq 0$. Evaluating the polynomial $p(h) = 3+3h+h^2$ at $h = 0$ gives $p(0) = 3$, so this 'difference quotient' approaches 3 as h approaches 0. Since $f(1) = 1$, The tangent line at $(1,1)$ is $y-1 = 3(x-1)$.

(8)

$$\frac{f(0+h)-f(0)}{h} = \frac{h^3-0^3}{h} == \frac{h^3}{h}$$

which equals h^2 for $h \neq 0$. Evaluating the polynomial $p(h) = h^2$ at $h = 0$ gives $p(0) = 0$, so this 'difference quotient' approaches 0 as h approaches 0. Since $f(0) = 0$, The tangent line at $(0,0)$ is $y-0 = 0(x-0)$.

(9)

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)-x)}{h} = \frac{h}{h}$$

which equals 1 for $h \neq 0$. Evaluating the polynomial $p(h) = 1$ at $h = 0$ gives $p(0) = 1$, so this 'difference quotient' approaches 1 as h approaches 0 for any value of x and $f'(x) = 1$. Since the curve $y = f(x)$ is a straight line with slope 1, we'd better hope that the slope of a curve computation reduces to the same slope as the line, and indeed it does.

(10)

$$\frac{f(x+h)-f(x)}{h} = \frac{2(x+h)+5-(2x+5)}{h} = \frac{2h}{h}$$

which equals 2 for $h \neq 0$. Evaluating the polynomial $p(h) = 2$ at $h = 0$ gives $p(0) = 2$, so this 'difference quotient' approaches 2 as h approaches 0 for any value of x and $f'(x) = 2$. Since the curve $y = f(x)$ is a straight line with slope 2, we'd better hope that the slope of a curve computation reduces to the same slope as the line, and indeed it does.

(11)

$$\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2-3x^2)}{h} = \frac{3x^2+6xh+3h^2-3x^2}{h} = \frac{6xh+3h^2}{h}$$

which equals $6x+3h$ for $h \neq 0$. Evaluating the polynomial $p(h) = 6x+3h$ at $h = 0$ gives $p(0) = 6x$, so this 'difference quotient' approaches $6x$ as h approaches 0 for any value of x and $f'(x) = 6x$. The curve $y = f(x)$ is a parabola, and it makes sense when $x > 0$ to the right of the downward pointing vertes, the slope increases as x increases.

(12)

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2-2(x+h)+3-(x^2-2x+3)}{h}$$

$$= \frac{x^2+2xh+h^2-2x-2h+3-x^2+2x-3}{h} = \frac{2xh+h^2-2h}{h}$$

which equals $2x + h - 2$ for $h \neq 0$. Evaluating the polynomial $p(h) = 2x + h - 2$ at $h = 0$ gives $p(0) = 2x - 2$, so this 'difference quotient' approaches $2x - 2$ as h approaches 0 for any value of x and $f'(x) = 2x - 2$.

(13)

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h}$$

which equals $3x^2 + 3xh + h^2$ for $h \neq 0$. Evaluating the polynomial $p(h) = 3x^2 + 3xh + h^2$ at $h = 0$ gives $p(0) = 3x^2$, so this 'difference quotient' approaches $3x^2$ as h approaches 0 for any value of x and $f'(x) = 3x^2$.

(14)

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 + (x+h)^2 - (x^3 - x^2)}{h}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 + x^2 + 2xh + h^2 - x^3 - x^2}{h} = \frac{3x^2h + 3xh^2 + h^3 + 2xh + h^2}{h}$$

which equals $3x^2 + 3xh + h^2 + 2x + h$ for $h \neq 0$. Evaluating the polynomial $p(h) = 3x^2 + 3xh + h^2 + 2x + h$ at $h = 0$ gives $p(0) = 3x^2 + 2x$, so this 'difference quotient' approaches $3x^2 + 2x$ as h approaches 0 for any value of x and $f'(x) = 3x^2 + 2x$.

These examples should show you three patterns.

1. The derivative of the sum of functions will equal the sum of the derivatives:

If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$. The aspects of the computation that always led to this did not have to do with the fact that the functions in the examples were polynomials.

2. The derivative of a constant multiple of a functions will equal the same constant multiple of its derivative:

If $f(x) = c(u(x))$ where c is a constant, then $f'(x) = c(u'(x))$. The aspects of the computation that always led to this did not have to do with the fact that the functions in the examples were polynomials.

3. The derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$ which comes from the binomial rule, $(x+h)^n = x^n + nx^{n-1}h + \ldots$.

More solutions on the following page!!

(15) The point-slope form of a line containing the point $(-2, 4)$ is $y-4 = m(x-(-2))$, where m is the slope. Using the definition of a tangent line, $m = f'(-2)$ where $f(x) = x^2$, so $f'(x) = 2x$. Therefore, $m = 2(-2) = -4$ and the equation of the tangent line is $y - 4 = -4(x - (-2))$. Note that we only need to be given the x-value, -2, from which we could compute the corresponding y-value, $f(-2) = 4$. The given equation $y - 4 = -4(x - (-2))$ corresponds to the form given in the notes, $y - f(a) = f'(a)(x - a)$ with $f(x) = x^2$ and $a = -2$. Depending on the situation, you may or may not wish to 'simplify' $(x - (-2))$ to $x + 2$ because the first form exhibits the key information more clearly, and from this point of view, the latter form is not a 'simplification'.

(16) The point-slope form of a line containing the point $(2, -2)$ is $y-(-2) = m(x-2)$, where m is the slope. Using the definition of a tangent line, $m = f'(2)$ where $f(x) = x^2-3x$, so $f'(x) = 2x - 3$. Therefore, $m = 2(2) - 3 = 1$ and the equation of the tangent line is $y-(-2) = 1(x-2)$. Note that we only need to be given the x-value, 2, from which we could compute the corresponding y-value, $f(2) = -2$. The given equation $y - (-2) = 1(x - 2)$ corresponds to the form given in the notes, $y - f(a) = f'(a)(x - a)$ with $f(x) = x^2 - 3x$ and $a = 2$. Again, whether you choose to 'simplify' $(y - (-2))$ to $y + 2$ depends on the situation. Using '$+c$' may save an arithmetic operation in a computation, but $-(-c)$ may have more clarity.