Diagnostic Test in Algebra and Trigonometry: Answers

Part I

1. Solve: $x - 2(3x - 1) = 7(3 - 2x)$.

Answer. For problems of this type, follow these steps:

1. $Remove\ \ parentheses\ :\qquad x - 6x + 2 = 21 - 14x$

2. $Bring\ like\ terms\ together:\ \ x - 6x + 14x = 21 - 2$

3. $Collect\ terms\ :\qquad\qquad 9x = 19$

4. $Divide\ by\ the\ coefficient\ of\ x\ :\qquad x = 19/9$

2. What is the equation of the line through the points (2,1) and (7,-5)?

Answer. For any two points on a line, the quotient of the difference in the y coordinates by difference in the x coordinates is always the same; this is the *slope* m of the line. Thus we have:

$$m = \frac{-5 - 1}{7 - 2} = -\frac{6}{5}$$

Now, the test for a general point (x, y) to be on the line is that the same computation, with this point and one of the given points, gives the same slope:

$$\frac{y - 1}{x - 2} = -\frac{6}{5}\ ,$$

which simplifies (by cross-multiplying) to:

$$5y - 5 = -6x + 12\ .$$

or $5y + 6x = 17$.

3. $Simplify:\qquad \dfrac{5 - \sqrt{3}}{4 + \sqrt{3}}$

Answer. By "simplify" we mean "get the square root out of the denominator". The clue for doing this is the identity $A^2 - B^2 = (A - B)(A + B)$. So, if we multiply and divide by $4 - \sqrt{3}$, we get

$$\frac{5 - \sqrt{3}}{4 + \sqrt{3}} \cdot \frac{4 - \sqrt{3}}{4 - \sqrt{3}} = \frac{20 - 9\sqrt{3} + 3}{16 - 3} = \frac{23 - 9\sqrt{3}}{13}$$

4. Solve

$$x + 2y = 11$$

$$8x - 3y = 31$$

Answer. There are two ways to solve simultaneous linear equations:
Elimination:

1. Make the coefficients in one of the variables the same in both equations. We get:

$$8x + 16y = 88$$

$$8x - 3y = 31$$

2. Subtract one equation from the other to eliminate a variable. Subtracting the second from the first, we get $19y = 57$, or $y = 3$.

3. Substitute this into the original first equation: $x + 2(3) = 11$, so $x = 5$, giving the answer (5,3).

Substitution:

1. Solve for one variable in terms of the other in one equation. From the first, we get $x = 11 - 2y$.

2. Substitute this in the second equation: $8(11 - 2y) - 3y = 31$.

3. Solve for that variable: $88 - 19y = 31$, so $y = 3$, and substituting that value in the equation $x = 11 - 2y$ gives $x = 5$.

5. Solve: $x^2 + 3x = 2(1 - x^2)$.

Answer. Remove parentheses, getting $x^2 + 3x = 2 - 2x^2$. Now bring all terms to one side: $3x^2 + 3x - 2 = 0$. Use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(3)(-2)}}{2(3)} = \frac{-3 \pm \sqrt{33}}{6} .$$

6. Solve
$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{9}{4}$$

Answer. First clear of fractions by multiplying both sides by $4(1 + x)(1 - x)$:

$$4(1 - x) + 4(1 + x) = 9(1 + x)(1 - x)$$

Now clear of parentheses and put like terms together:

$$4 - 4x + 4 + 4x = 9 - 9x^2$$

leading to $9x^2 - 1 = 0$, or $x = \pm 1/3$.

7. Let

$$f(u) = \frac{u+1}{u^2+1}$$

For $u(x) = 2x - 1$, what is $f(u(x))$?

Answer. Substitute $2x - 1$ for u in the expression for x, and simplify:

$$f(u(x)) = \frac{(2x-1)+1}{(2x-1)^2+1} = \frac{2x}{4x^2 - 4x + 1 + 1} = \frac{x}{2x^2 - 2x + 1}$$

8. Let

$$y = \frac{1-x}{1+x}$$

Write x in terms of y.

Answer. Follow the usual procedure: clear of fractions, clear of parenthesis, collect terms in x on one side, then solve for x:

$$y(1+x) = 1 - x \ , \qquad y + xy = 1 - x \ , \qquad xy + x = 1 - y$$

resulting in

$$x = \frac{1-y}{1+y}$$

9. *Solve* :

$$\sqrt{2x-1} = \frac{x+1}{2} - 6$$

Answer. Clear of fractions, and then square to eliminate the radical:

$$2\sqrt{2x-1} = x - 11 \qquad \textit{squares to} \qquad 4(2x-1) = x^2 - 22x + 121$$

Now collect terms:

$$8x - 4 = x^2 - 22x + 121 \qquad \textit{becomes} \qquad x^2 - 30x + 125 = 0$$

This factors to $(x-5)(x-25) = 0$, so $x = 5$ or $x = 25$. When we squared the expression, we introduced the possibility of *extraneous* roots, so we have to check both possible answers. For $x = 5$, the original expression becomes 3 = 3-6 , which is false. For $x = 25$, the expression becomes 7=13-6, which is true. Thus the only answer is $x = 25$. Notice that (in Calculus) $\sqrt{a}$ always means the positive square root of a.

10. If the hypotenuse of a right triangle is of length 5 and one leg is of length 2, what is the length of the other leg?

Answer. The Pythagorean theorem is one of the most used facts of geometry. It says that "the square on the hypotenuse is the sum of the squares on the two sides." That is, $a^2 + b^2 = c^2$ for a right triangle, where c is the length of the hypotenuse. Letting $b = 2,\ c = 5$, we solve for a: $a^2 + 2^2 = 5^2$, which leads to $a^2 = 25 - 4$, so $a = \sqrt{21}$.

11. If $\sin\theta = 3/5$, what is $\cos\theta$?

Answer. From the Pythagorean theorem, and the definition of the trigonometric functions, we obtain the identity $\sin^2\theta + \cos^2\theta = 1$, so in our case, $(3/5)^2 + \cos^2\theta = 1$. Solving, we obtain $\cos\theta = \pm\sqrt{1 - (3/5)^2} = \pm 4/5$.

12. Verify the identity:

$$\frac{\cos x + \sin x}{\cos x - \sin x} = \frac{1 + \sin(2x)}{\cos(2x)}$$

Answer. Several identities of trigonometry are important in Calculus; the one in problem 11 most of all, but also the summation identities:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\ , \quad \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Letting $\alpha = \beta = x$ on the right hand side, we get

$$\frac{1 + 2\sin x\cos x}{\cos^2 x - \sin^2 x} = \frac{\sin^2 x + 2\sin x\cos x + \cos^2 x}{(\cos x + \sin x)(\cos x - \sin x)} = \frac{(\sin x + \cos x)^2}{(\cos x + \sin x)(\cos x - \sin x)}$$

giving us the left hand side once we cancel the common factor $\sin x + \cos x$.

Part II

1. *Answer* $$\frac{-2}{2/5} + \frac{3}{2/7} - \frac{1/2}{2/9} = -2\frac{5}{2} + 3\frac{7}{2} - \frac{1}{2}(\frac{9}{2}) = \frac{-20 + 42 - 9}{4} = \frac{13}{4}$$

2. *Solve* $$\frac{2x(x+2)^{1/2} - (3x+1)(x+2)^{-1/2}}{x+2} = 0$$

Answer. Multiply numerator and denominator by $(x+2)^{1/2}$ and simplify the expression. We get:

$$\frac{2x(x+2) - (3x+1)}{(x+2)^{3/2}} = \frac{2x^2 + x - 1}{(x+2)^{3/2}}$$

Now solve by setting the numerator equal to zero and use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1+8}}{2(2)} = -1 \quad or \quad 1/2 \ .$$

3. *Solve* $\qquad \sin x + 2 - \sin^2 x + \cos^2 x = 0$

Answer. Make this an equation in $\sin x$ alone:

$$\sin x + 2 - \sin^2 x + 1 - \sin^2 x = 0 \quad or \quad -2\sin^2 x + \sin x + 3 = 0$$

Now use the quadratic formula:

$$\sin x = \frac{-1 \pm \sqrt{1+24}}{-4} = \frac{1}{4} \pm \frac{5}{4} = \ \frac{3}{2} \ \text{ or } \ -1 \ .$$

($3/2 > 1$, so cannot be a sine, so the answer is $\arcsin(-1) = 3\pi/2$.

4. *Solve* $\qquad \sqrt{x+5} + \sqrt{x-1} = 11 \ .$

Answer. We move one of the square roots to the other side, and square:

$$x + 5 = (11 - \sqrt{x-1})^2 = 121 - 22\sqrt{x-1} + x - 1$$

$$22\sqrt{x-1} = 115$$

so $x - 1 = (115/22)^2 = 27.32$, giving $x = 28.32$.

5. $Solve for \ \ x \ \ and \ \ y \ : \ \ (x-1)^2 = y - 1 \ , \qquad 2x + y = 11 \ .$

Answer. From the second equation we get $y = 11 - 2x$. Substitute this in the first, getting

$$x^2 - 2x + 1 = 10 - 2x \qquad or \qquad x^2 = 9$$

The solutions could be $x = 3, -3$. For $x = 3$, we get $y = 5$, and for $x = -3$, $y = 17$. Check that both pairs are solutions.

6. Find the equation of the line through (2,1) which is perpendicular to the line through the points (5,8) and (3,-1).

Answer. . Two lines are perpendicular if the product of their slopes is -1. The slope of the line through (5,8) and (3,-1) is (8-(-1))/(5-3) = 9/2. Thus the slope of the line sought is -2/9. The equation then is

$$\frac{y-1}{x-2} = -\frac{2}{9} \quad or \quad 9y + 2x = 13$$

7.
$$\frac{x^3 + 10x^2 + 3x - 54}{x^2 + 7x - 18}$$

is a polynomial. Why? Find the polynomial.

Answer. The denominator factors into $(x-2)(x+9)$, so has 2,-9 as roots. These are also roots of the numerator (to be checked) and that is why the quotient is a polynomial. We find the quotient, $x+3$, by long division.

8.
$$\frac{x-7}{x+13} = 1 - \frac{a}{x+13} \quad . \quad a = \ ?$$

Answer. When we multiply through by the denominator, $x + 13$, the variable x conveniently drops out:

$$x - 7 = x + 13 - a \quad so \quad a = 20$$

This is a good trick to remember to simplify any expression of this form:

$$\frac{x+\alpha}{x+\beta} = \frac{(x+\beta) + (\alpha - \beta)}{x+\beta} = 1 + \frac{\alpha - \beta}{x + \beta}$$

9. For what values of x between 0 and $\pi/2$ (angle measured in radians) is $\sin x > \cos x$?

Answer. Since both sides are positive in this range, the inequality is the same as $\tan x > 1$. Now remembering the graph of $y = \tan x$, we have the answer: $\pi/4 < x < \pi/2$.

10. Consider the expression

$$f(x) = \frac{x(x-1)}{(x+1)(x-2)}$$

a) If $x < -1$, what is the sign of $f(x)$? **Answer:** positive.
b) If $-1 < x < 0$, what is the sign of $f(x)$? **Answer:** negative.
c) If $x > 2$, what is the sign of $f(x)$? **Answer:** positive.

d) How many times does the sign of $f(x)$ change as x ranges from -2 to 4? **Answer:** 4 times.

To answer this kind of question, notice that a factor of the form $x - a$ changes sign at a, but the sign of all other factors remains the same. Thus since there are four such terms, there are four such sign changes. But, be careful! If we have a factor of the form $(x - a)^n$, then if n is even, there is no sign change at a. FOr example, for

$$f(x) = \frac{x(x-1)}{(x+1)^2(x-2)}$$

there are only three sign changes, for $(x + 1)^2$ does not change sign as at -1.