CHAPTER 4

Integration

§4.1. Antiderivatives

The basic idea of Newton's about dynamics is this: if we know the state of a system at a particular time, and we know the laws of change, then we can predict the state at any future time. This is embodied in his first law of motion which says that, in the absence of external forces, an object in motion will continue its motion in the same direction with the same speed. Put another way, if acceleration is zero, then velocity is constant; and yet another way: if $dv/dt = 0$, then $v(t) = v(0)$ for all t. We have already seen this in Chapter 2 (as theorem 2.5) :

Theorem 4.1 *Suppose that f is differentiable in an interval, and has derivative zero everywhere. Then f is constant.*

As a consequence, we have

Proposition 4.1 *If two functions have the same derivative, they differ by a constant.*

For suppose f and g are the two functions, and $f' = g'$. We can apply theorem 4.1 to $h = f - g$: $h' = f' - g' = 0$, so $h(x) = C$, some constant. Then $f(x) = g(x) + C$.

Definition 4.1 *Given a function f, any function F such that $F' = f$ is an antiderivative or indefinite integral, or just integral of f. Any integral is denoted $\int f(x)dx$.*

We emphasize that any two integrals of a given function differ by a constant. So, for example, we know that if $f(x) = x^2$, then $f'(x) = 2x$, so x^2 is an integral of $2x$, and therefore any integral of $2x$ is of the form $x^2 + C$, for some constant C. We indicate this by writing $\int 2xdx = x^2 + C$. Now, the formulae of the differential calculus lead to the formulae for finding integrals, although not always so easily, as we shall see. This process of finding integrals is called *integration*. For example, since the derivative of a sum is the sum of the derivatives, then the integral of a sum is the sum of the integrals. Since we differentiate x^n

by multiplying by the exponent and reducing the exponent by 1, we integrate x^n by reversing the process: increase the exponent by 1, and divide by the new exponent. To summarize:

Proposition 4.2

a) Let f be a given function, and a a number. Then $\int af(x)dx = a\int f(x)dx$.

b) Let f and g be given functions. Then $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$

c) $\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1.$

Of course, the exclusion $n=-1$ is necessary, for in this case the right hand side doesn't make sense.

Example 4.1 Find the integral of $f(x) = x^4 - 3x^2 + x - 1$.

We integrate term by term, using Proposition 4.2c for each term:

$$\int f(x)dx = \frac{x^5}{5} - 3\frac{x^3}{3} + \frac{x^2}{2} - x + C \tag{4.1}$$

$$= \frac{1}{5}x^5 - x^3 + \frac{1}{2}x^2 - x + C \tag{4.2}$$

Example 4.2 $\int (4x^{-3}+x^2)dx = 4\left(\frac{x^{-2}}{-2}\right) + \frac{x^3}{3} + C = -2x^{-2} + \frac{x^3}{3} + C$

Example 4.3 A function $f(x)$ has the derivative $f'(x) = x^2 + x^{-2}$, and the value at $x=2$ is 5. What is the function?

First we find the general function with the given derivative by integrating term by term:

$$f(x) = \frac{1}{3}x^3 - x^{-1} + C \tag{4.3}$$

Now we substitute the given values, and solve for C:

$$5 = \frac{1}{3}2^3 - \frac{1}{2} + C \tag{4.4}$$

giving $C = 17/6$. Thus the desired function is

$$f(x) = \frac{1}{3}x^3 - x^{-1} + \frac{17}{6} \tag{4.5}$$

Proposition 4.2 shows us how to find integrals of polynomials. The differentiation formulae for the trigonometric functions also lead to integration formulae for these functions.

Proposition 4.3

a) $\int \cos x dx = \sin x + C$

b) $\int \sin x dx = -\cos x + C$

So far, we can only integrate functions by, so to speak, reading a table of derivatives in the reverse direction. For example, we also know that $\int \sec^2 dx = \tan x + C$ and $\int \sec x \tan x dx = \sec x + C$, but we don't yet know the integral of $\sec x$, or for that matter $\tan x$ nor x^{-1}. Finding integrals in general is a quite complicated process, and as this course proceeds we will study the various techniques of integration.

The first, and most useful of these techniques is that of *substitution*. This is the integration form of the chain rule. It is most conveniently stated in terms of differentials.

Proposition 4.4 *Given variables u and x; suppose we know that $u = u(x)$ is a function of x. Suppose their differentials are related by*

$$f(x)dx = g(u)du \tag{4.6}$$

for some functions f and g. Then

$$\int f(x)dx = \int g(u)du + C\ . \tag{4.7}$$

To see this, let $G(u) = \int g(u)du$. Then, treating G as a function of x by the substitution $u = u(x)$, we have

$$\frac{dG}{dx} = \frac{dG}{du}\frac{du}{dx} \tag{4.8}$$

by the chain rule. But $dG/du = g(u)$, so $\dfrac{dG}{dx} = g(u)\dfrac{du}{dx} = f(x)$ by the relation 4.6. Thus $G(u(x))$ is an integral of $f(x)$, so differs from $F(x)$ by a constant.

This explains in part the notation for the integral: we should be thinking that it is the differential $f(x)dx$ which we are integrating, rather than the function. For when we change variables by substitution, it is the entire differential which we must consider.

Example 4.4 $\int (5x-3)^5 dx = ?$

Since we don't want to multiply $5x-3$ by itself 5 times so we can use Proposition 4.2, we instead introduce the variable $u = 5x-3$. Then $du = 5dx$, or $dx = (1/5)du$, so

$$(5x-3)^5 dx = \frac{1}{5}u^5 du\ . \tag{4.9}$$

We now apply the power rule to the right hand side:

$$\int \frac{1}{5}u^5 du = \frac{1}{5}\cdot\frac{1}{6}u^6 + C \tag{4.10}$$

and then replace u by its expression $5x-3$ as a function of x:

$$\int (5x-3)^5 dx = \frac{1}{30}(5x-3)^6 + C\ . \tag{4.11}$$

Example 4.5 $\int x(x^2+1)^3 dx = ?$

To integrate by substitution, always let u be what is inside the most complicated part. Here we want to let $u = x^2+1$. Then $du = 2xdx$, so we can replace the differential to be integrated as follows:

$$x(x^2+1)^3 dx = \frac{1}{2}(x^2+1)^3(2xdx) = \frac{1}{2}u^3 du\ . \tag{4.12}$$

Then

$$\int x(x^2+1)^3 dx = \frac{1}{2}\int u^3 du = \frac{1}{2}\cdot\frac{1}{4}u^4 + C = \frac{1}{8}(x^2+1)^4 + C\ . \tag{4.13}$$

Example 4.6 $\int \cos^2(2x+1)\sin(2x+1)dx = ?$

Let $u = \cos(2x+1)$. Then $du = -2\sin(2x+1)dx$. This substitution is effective:

$$\int \cos^2(2x+1)\sin(2x+1)dx = -\frac{1}{2}\int u^2 du = -\frac{1}{2}\frac{u^3}{3} + C = -\frac{1}{6}\cos^3(2x+1) + C\ . \tag{4.14}$$

§4.2. Separation of Variables

A *first order differential equation* is a relation among the variables $y,\ y',\ x$. For example:

$$y' = x^2+1\ , \quad xy' = y + 2x^3\ , \quad (y')^2 + \sin^2 x = 1\ . \tag{4.15}$$

A *solution* of a differential equation is a function $f(x)$ such that, if we let $y = f(x),\ y' = f'(x)$ in the equation, we get an identity. So, for the above examples, we can check that the following are solutions, respectively:

$$y = \frac{1}{3}x^3 + x + 5\ , \quad y = x^3 + 2x\ , \quad y = \sin x\ . \tag{4.16}$$

In general, it is difficult to find solutions to differential equations, but in one special case, thanks to proposition 4.4, it is not so hard. That proposition says that if we know that the differentials $f(y)dy$ and $g(x)dx$ are equal, then their integrals differ by constant. In that proposition we assumed the prior knowledge that y is a function of x, but the technique still works without that assumption. In fact, it comes out as a conclusion!

So, suppose that, upon replacing y' by dy/dx, we can rewrite the differential equation as an equation of differentials:

$$f(y)dy = g(x)dx \tag{4.17}$$

then we can solve by integrating both sides.

Example 4.7 Solve the differential equation $y' = \dfrac{x^2}{y}$.

We rewrite this as $dy/dx = x^2/y$, which can be rewritten in differential form as $ydy = x^2dx$. Now, integrate both sides:

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C \tag{4.18}$$

or

$$y^2 = \frac{2}{3}x^3 + C \tag{4.19}$$

(Since C represents a generic constant, so does $2C$, so we can again call it C). Thus the solution is

$$y = \sqrt{\frac{2}{3}x^3 + C}\ , \tag{4.20}$$

and this is the general solution.

Example 4.8 Find the particular solution of the differential equation $y' = \dfrac{x^2}{y}$ such that $y = 7$ when $x = 3$.

We follow the above argument to the equation

$$y^2 = \frac{2}{3}x^3 + C \tag{4.21}$$

Here we use the condition $y = 7$ when $x = 3$ to identify C:

$$7^2 = \frac{2}{3}3^3 + C \quad or \quad 49 = 18 + C \tag{4.22}$$

so that $C = 31$. Then the solution is

$$y = \sqrt{\frac{2}{3}x^3 + 31}\ . \tag{4.23}$$

Example 4.9 In the study of an epidemic of an airborne disease which is reinforced by prolonged exposure to those infected, a first working hypothesis may be that the rate of spread of the disease is proportional to the amount of interaction among those infected, which is in turn proportional to the square of the population of infected. We then ask, how long will it take, unless some action is taken, for the entire population to be infected? To work through an example, suppose the rate of spread (in units of population per day) is equal to one thousandth the square of the population. If the original infected population is 100, how many will be infected at some future time t?

Let $P(t)$ represent the population of infected people at time t. The given law of change is

$$\frac{dP}{dt} = .001P^2 \tag{4.24}$$

We can rewrite this in the form

$$P^{-2}dP = .001dt \tag{4.25}$$

Integrating, we find

$$-P^{-1} = .001t + C\ . \tag{4.26}$$

Now when $t = 0$, $P = 100$, so $C = -.01$, so we have

$$-P^{-1} = .001t - .01\ , \tag{4.27}$$

or $P(t) = (.01 - .001t)^{-1}$. So, for example, after 5 days the infected population is $(.01 - .005)^{-1} = 200$, and after 8 days it is 500. Worst of all, in ten days, P is infinite: everyone is infected, no matter how large the original healthy population was.

Example 4.10 Suppose a ball is thrown upward at an initial velocity of 128 ft/sec. How high does it go?

Let s, v, a represent distance traveled (measured upwards, with the surface of the earth at $s = 0$), velocity, acceleration. The acceleration due to gravity is

$$\frac{dv}{dt} = a = -32\ \ \text{ft}/\text{sec}^2\ . \tag{4.28}$$

We integrate and conclude that $v = -32t + C$, for some constant C. Now, since $v = 128$ at time $t = 0$, we have

$$\frac{ds}{dt} = v = -32t + 128 \tag{4.29}$$

and integrating again we obtain $s = -16t^2 + 128t + C$. Since $s = 0$ when $t = 0$, we must have $C = 0$, so

$$s = -16t^2 + 128t \tag{4.30}$$

Now, at the maximum height, the velocity is zero. Solving 4.29 for $v = 0$, we have $t = 128/32 = 4$ seconds at the high point. Putting this value of t in 4.30, we obtain the answer

$$s = -16 \cdot 4^2 + 128(4) = 256\ . \tag{4.31}$$

Example 4.11 In the above problem, the time it takes to attain maximum height is calculated in order to find that height, but is otherwise irrelevant. In fact, by eliminating the variable t, we can find a more direct relation between distance traveled and velocity. Once again, we consider the variables s, v, a, t, and relate their differentials by

(4.32) $$ds = vdt\ , \qquad dv = adt\ .$$

But now, we eliminate dt by multiplying the second equation by v:

(4.33) $$vdv = avdt = ads$$

from which we conclude, by integrating

(4.34) $$\frac{1}{2}v^2 = \int ads$$

which is useful if a is a function of distance alone. In the case of the above problem a is constant, so 4.34 becomes $(1/2)v^2 = as + C$. Setting $s = 0$ as the initial position, and v_0 the initial velocity, we find $C = (1/2)v_0^2$, giving the relation

(4.35) $$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = as$$

Now, in the preceding problem $a = -32$, $v_0 = 128$ and $v = 0$ at the maximum height, so we get $-(1/2)(128)^2 = -32s$; solving for s gives $s = 256$.

Example 4.12 An automobile traveling at a speed of 60 mph (88 ft/sec) decelerates at a rate of 12 ft/sec^2. How far does it travel before it stops?

Here $a = -12$, $v_0 = 88$ and $v = 0$ when it stops. Putting these data into 4.35:

(4.36) $$-\frac{1}{2}(88)^2 = -12s$$

giving $s = 322.67$ ft.

Example 4.13 Now, Newton's second law, $F = ma$ says that the acceleration of a body of mass m in motion is proportional to the force F applied to it. If those forces are spatial; that is, functions of s alone, then, by multiplying equation 4.34 by m we get

(4.37) $$\frac{1}{2}mv^2 = \int F(s)ds$$

which is a way of stating the law of conservation of energy: the change in kinetic energy of the moving object (the left hand side) is equal to the work done by the force (the right hand side).

Example 4.14 To illustrate this observation, consider a rocket sitting on the surface of a planet of mass M and radius R. With what initial velocity v_0 should the rocket be propelled so as to escape the gravitational field of the planet?

According to Newton's law of universal gravitation, the force F of gravity is given by

$$F = -G\frac{mM}{s^2} \tag{4.38}$$

where G is a universal constant, m is the mass of the rocket, and s is the distance of the rocket from the center of the planet. In particular, from $F = ma$, we obtain

$$a = -G\frac{M}{s^2}\ , \tag{4.39}$$

and equation 4.37 becomes

$$\frac{1}{2}v^2 = -GM\int s^{-2}ds = GMs^{-1} + C\ . \tag{4.40}$$

At liftoff, $s = R$ and $v = v_0$, and we find

$$C = \frac{1}{2}v_0^2 - GMR^{-1}\ , \tag{4.41}$$

and thus the velocity and distance of the rocket at any future time satisfy the relation

$$v^2 = \frac{2GM}{s} + v_0^2 - \frac{2GM}{R}\ . \tag{4.42}$$

The rocket will escape the planet if this is always positive, so we must insure that $v_0 \geq \sqrt{2GM/R}$. In particular, if the planet is earth, then $GM/R^2 = g = 32$ ft/sec^2, and $R = 3900$ miles. Converting to miles and hours, we find that the initial velocity must be approximately 24,750 mph for the rocket to escape the earth.

§4.3. Area and Definite Integrals

In the preceding sections we followed the thinking and methods of Newton on integrations. As in Chapter 1, we now turn to Leibniz for his ideas on the subject. For Leibniz, integration is a method of accumulation; or of approximation and accumulation, to be more precise. To fix the ideas, we start with the calculation of area under a curve. Suppose that $y = f(x)$ is a non-negative function defined on the interval $[a,b] = \{x; a \leq x \leq b\}$. We want to find the area of the region bounded by the curve, the x-axis and the lines $x = a$, $x = b$. If we pick some point c between a and b, then we know that the area under the curve $y = f(x)$ and over the interval $[a,b]$ is the sum of the areas over the intervals $[a,c]$ and $[c,b]$. In fact, if we cut the interval $[a,b]$ into any number of little intervals, the area of the whole is the sum of the areas over all the little intervals. Now, if the little intervals are small enough, and if the function is continuous, then the area over that interval is approximately equal to the area of the column over that interval of height $f(c)$, for some c in the interval. See figure 4.1 for a graphic of this process.

Leibniz' notation for this approximation is

$$\sum_a^b f(x)\Delta x \tag{4.43}$$

Figure 4.1

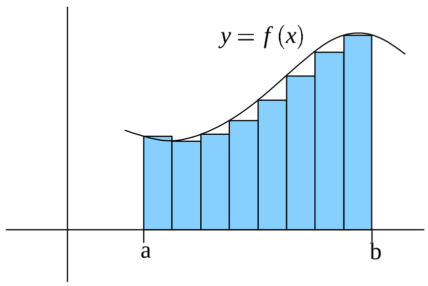
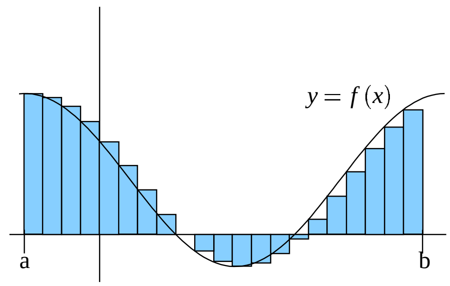


Figure 4.2



where Δx represents the length of the base of a typical column, $f(x)$ its height, and the symbol $\sum$ indicates that we add all these together. Here is where Leibniz takes the great leap: suppose the little intervals are of infinitesimal length; that is Δx is the infinitesimal dx. Then this "approximation" is precise, and we obtain the actual area of the figure, denoted by

$$\int_a^b f(x)dx \; . \tag{4.44}$$

Now, there are many processes besides that of calculating area (as we shall see in the next chapter) which have this accumulative property: that the whole is the sum of its parts, and we can calculate the value on the whole by adding the values of all of its parts. Thus, Leibniz goes on to discuss this process for any function $y = f(x)$.

Definition 4.2 *Let $y = f(x)$ be a function defined on the interval $[a,b]$. The definite integral is defined as follows. A partition of the interval is any increasing sequence*

$$\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\} \tag{4.45}$$

of points in the interval. The corresponding approximating sum is

$$\sum_1^n f(x_i')\Delta x_i \tag{4.46}$$

where Δx_i is the length $x_i - x_{i-1}$ of the ith interval, x_i' is any point on that interval, and $\sum$ indicates that we add all these products together (see figure 4.2). If these approximating sums approach a limit as the partition becomes increasingly fine (the lengths of the little intervals go to zero), this limit is the definite integral of f over the interval $[a,b]$, denoted

$$\int_a^b f(x)dx \; . \tag{4.47}$$

This definition raises some serious questions. Let's return to the calculation of area. Surely, as we have observed, if the interval Δx is small enough, then the term $f(x')\Delta x$ is within a very small error of the actual area over the interval. But now we add together a large number of these approximations, and so the errors add. How do we know that the accumulated error is not fatal? This issue also took several centuries to be satisfactorily resolved; for us it is enough to know that it does work:

Theorem 4.2 *If $y = f(x)$ is a continuous function on the interval $[a,b]$, then the integral $\int_a^b f(x)dx$ exists. If f is a nonnegative function, this integral is the area under the curve.*

The following properties of the definite integral follow easily from the definition.

Theorem 4.3 *Suppose that f and g are continuous functions on the interval $[a,b]$.*

a) $\int_a^b Af(x)\,dx = A\int_a^b f(x)\,dx$

b) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

If c is any point in $[a,b]$:

c) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

if $f(x) \geq g(x)$ for all x in $[a,b]$, then

d) $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

In particular, the definite integral of a positive function is positive.

Now the process described in definition 4.2 is almost impossible to carry out in practice (we shall provide examples in the next section). Fortunately, definite integrals can be calculated using the indefinite integral, as we now show.

Theorem 4.4 (Fundamental Theorem of the Calculus, I) *Suppose that $y = f(x)$ is a continuous function on the interval $[a,b]$. If F is any indefinite integral of f, then*

$$\int_a^b f(x)dx = F(b) - F(a) \;. \tag{4.48}$$

We see this by looking at the accumulation process dynamically: we calculate the definite integral by accumulating from left to right. For any x in the interval, let $\mathrm{I}(x)$ be the value of the definite integral from a to x. Now, if we go slightly further, say, to $x+\Delta x$, then, by the defining process, $f(x)\Delta x$ is an approximation to the increment in I:

$$\Delta\mathrm{I} = \mathrm{I}(x+\Delta x) - \mathrm{I}(x) = f(x)\Delta x \quad \text{approximately} \;. \tag{4.49}$$

Now, moving to a differential increment, we obtain the equality

$$d\mathrm{I} = f(x)dx \tag{4.50}$$

so I is an indefinite integral of f. Thus $\mathrm{I}(x) = F(x) + C$, for some constant C. Since $\mathrm{I}(a) = 0$,

$$0 = F(a) + C\,, \quad so \quad C = -F(a) \tag{4.51}$$

so that $\mathrm{I}(x) = F(x) - F(a)$ for all x. In particular, evaluating at $x = b$ gives us 4.48. In the above argument, we used I to represent $\int_a^x f(t)dt$. Putting this in equation 4.49 gives the second version of the fundamental theorem of the Calculus:

Theorem 4.5 (Fundamental Theorem of the Calculus, II)
$$\frac{d}{dx}\int_a^x f(t)dt = f(x)\ .$$

If we divide both sides of equation 4.49 by Δx, we get

$$\frac{\Delta \mathrm{I}}{\Delta x} = f(x) \quad \text{approximately}\ . \tag{4.52}$$

Then, in the limit $d\mathrm{I}/dx = f(x)$, but $\mathrm{I} = \int_a^x f(t)dt$. So, to calculate a definite integral $\int_a^b f(x)dx$, follow these steps:

1. Find an indefinite integral F for f;
2. Evaluate $F(b) - F(a)$.

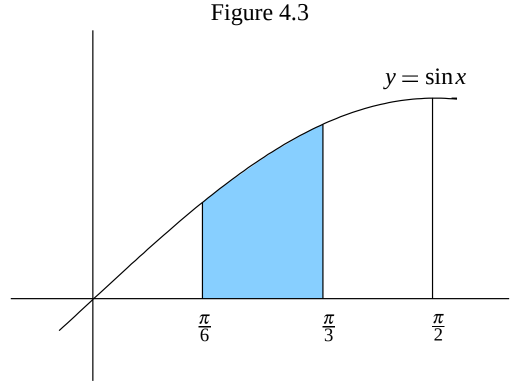
In actual calculations it is customary and convenient to use the notation $F(x)|_a^b$ as an intermediary between these two steps.

Example 4.15 $\displaystyle\int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = 8$

In the first step we found the indefinite integral $x^3/3$, and in the second, evaluated it at 3 and 1, and took the difference.

Example 4.16 Find the area under the curve $y = \sin x$ between $x = \pi/6$ and $x = \pi/3$ (see Figure 4.3).

Figure 4.3



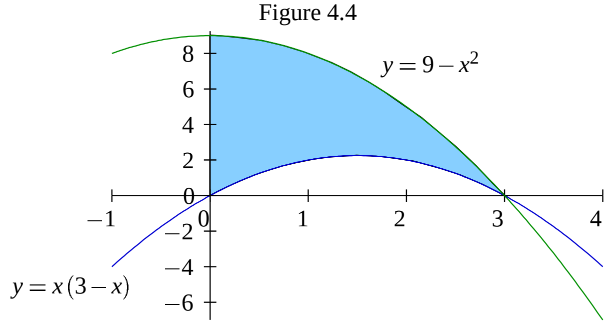
$$\int_{\pi/6}^{\pi/3} \sin x dx = (-\cos x)\Big|_{-\pi/6}^{\pi/3} = -\cos\left(\frac{\pi}{6}\right) - \left(-\cos\left(\frac{\pi}{3}\right)\right) = \frac{\sqrt{3}-1}{2} \tag{4.53}$$

In general, the way to find the area of a region is this. Sketch the region under consideration. Choose a direction in which to accumulate the area. Write down the expression for the differential increment in area: $dA = L(x)dx$, where dx is an infinitesimal increment in the direction of accumulation, and $L(x)$ is the length of the column over that increment.

Example 4.17 Find the area of the region bounded by the line $x = 0$ and the curves $C_1 : y = 9 - x^2$, $C_2 : y = x(3-x)$.

The sketch of this region is given in figure 4.4.

Figure 4.4



Here we will accumulate area in the direction of the x-axis from $x = 0$ to $x = 3$. At a particular value of x, the length of the column is the difference of the values of y on the two curves. Express this in terms of x:

(4.54) $$dA = [(9 - x^2) - x(3-x)]dx$$

Thus, the area is given by the integral

(4.55) $$\begin{aligned}\int_0^3 [(9-x^2) - x(3-x)]dx &= \int_0^3 \left[9 - x^2 - 3x + x^2\right] dx = \int_0^3 (9-3x)\, dx \\ &= \left[9x - \frac{3}{2}x^2\right]\Big|_0^3 = \left[9\,(3) - \frac{3}{2}3^2\right] - 0 = \frac{27}{2}\end{aligned}$$

Example 4.18 Find the area of the region bounded by the curves $x = -y^2$, $x = y^2$, $y = -2$ and $y = 2$.

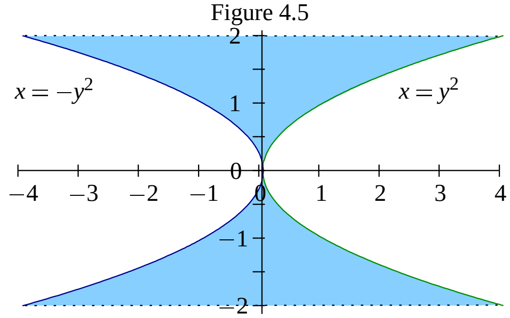
See figure 4.5 for the sketch. Here we choose to accumulate area in the y direction. The infinitesimal increment at a particular value of y is

(4.56) $$dA = \left(y^2 - \left(-y^2\right)\right) dy = 2y^2 dy.$$

Thus

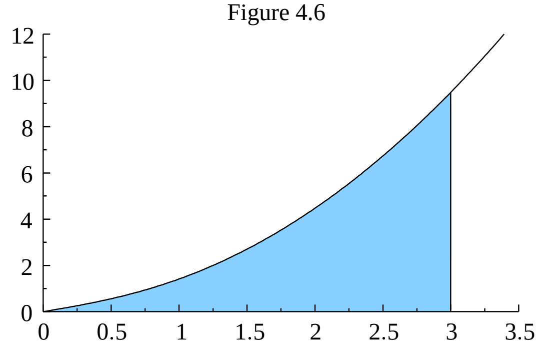
(4.57) $$A = \int_{-2}^{2} 2y^2 dy = \frac{2}{3}y^3\Big|_{-2}^{2} = \frac{2}{3}2^3 - \left(\frac{2}{3}(-2)^3\right) = \frac{32}{3}\,.$$

Figure 4.5



Example 4.19 Find the area between the curve $y = x\sqrt{x^2+1}$ and the x-axis, from $x = 0$ to $x = 3$ (see Figure 4.6).

Figure 4.6



Here $dA = x\sqrt{x^2+1}dx$, so the area is $\int_0^3 x\sqrt{x^2+1}dx$. We integrate by using the substitution $u = x^2+1,\ du = 2xdx$. When $x = 0,\ u = 1$, and when $x = 3,\ u = 10$. The area is

$$\int_0^3 x\sqrt{x^2+1}dx = \frac{1}{2}\int_1^{10} u^{1/2}du = \frac{1}{2}\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{1}{3}(10\sqrt{10}-1) \tag{4.58}$$

Notice, that when we make a substitution in a definite integral, we also replace the limits of integration by the values of the new variable at the endpoints. In this way the computation is easier than resubstituting back at the end.

Example 4.20 An object moves along the x-axis so that its velocity at time t is $v(t) = t(t^2-1)^5$. If the object is at the origin at time $t = 0$, at what point is it at time $t = 1$? At time $t = 2$?

Notice that the velocity is negative until $t = 1$, so the object starts moving to the left, but at $t = 1$ turns around and moves to the right. No matter; its position is still given by the definite integral:

$$x(1) = \int_0^1 vdt = \int_0^1 t(t^2-1)^5dt \tag{4.59}$$

Make the substitution $u = t^2 - 1,\ du = 2dt$. When $t = 0,\ u = -1$, and when $t = 1,\ u = 0$, so

$$x(1) = \frac{1}{2}\int_{-1}^{0} u^5 du = \frac{1}{12}u^6\Big|_{-1}^{0} = -\frac{1}{12} \tag{4.60}$$

For $t = 2$, we have $u = 3$, so

$$x(2) = \frac{1}{2}\int_{-1}^{3} u^5 du = \frac{1}{12}u^6\Big|_{-1}^{3} = \frac{1}{12}(3^6 - 1)\ . \tag{4.61}$$

§4.4. Summation and the Definite Integral

In this section we shall make the definition of the definite integral precise, and shall do some computations directly from the definition. The purpose here is to emphasize that the definite integral represents a process of accumulation, and to introduce and work with the notation for summation.

Let n be a positive integer, and $a(k)$ a rule that assigns a number to each integer between 1 and n. For example:

$$a(k) = 1\ for\ all\ k\ ; \quad or \quad a(k) = 2k+3\ ; \quad or \quad a(k) = k^2\ ; \quad or \quad a(k) = \frac{3}{k}\ , \tag{4.62}$$

for k running from 1 to $n = 100$. The sum of all these numbers is denoted

$$\sum_{k=1}^{n} a(k) = a(1) + a(2) + \cdots + a(n)\ . \tag{4.63}$$

This sum can sometimes be easily expressed as a formula in n; more often this is difficult, or impossible. For example, the sums in the first three cases are, respectively

$$n\ ; \qquad n(n+4)\ ; \qquad \frac{n(n+1(2n+1)}{6}\ , \tag{4.64}$$

whereas a formula for the sum in the fourth case is not known. These formulas are not easy to come by, and usually rely on clever tricks. Here are the cases we will need for application to integration.

Proposition 4.5

a) $Z(n) = \sum_1^n 1 = 1 + 1 + \cdots + 1 = n$

b) $U(n) = \sum_1^n k = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

c) $S(n) = \sum_1^n k^2 = 1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d) $K(n) = \sum_1^n k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

4.4 is just the obvious statement that the sum of n ones is n. To show 4.4 we start with the observation that

$$(k+1)^2 = k^2 + 2k + 1 \tag{4.65}$$

Now add these all together from $k = 1$ to $k = n-1$. On the left hand side we get the sum of all the squares from 2^2 to n^2, or $S(n) - 1^2$. The sum of the first terms on the right hand side is $S(n-1)$, of the second is $2U(n-1)$ and the last contributes $n-1$. Thus

$$S(n) - 1 = S(n-1) + 2U(n-1) + n - 1 \quad \textit{or} \quad 2U(n-1) = S(n) - S(n-1) - n\,. \tag{4.66}$$

Now add $2n$ to both sides, and remember that $S(n) - S(n-1) = n^2$:

$$2U(n) = n^2 + n\,, \tag{4.67}$$

which gives 4.4. For 4.4 (and subsequently 4.4, and all higher powers), we use the same idea. Start with

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1 \tag{4.68}$$

Add these all together from $k = 1$ to $k = n-1$, and calculate each term as above to get:

$$K(n) - 1 = K(n-1) + 3S(n-1) + 3U(n-1) + n - 1\,. \tag{4.69}$$

We know $U(n-1)$ from a), and $K(n) - K(n-1) = n^3$, so we get:

$$3S(n-1) = n^3 - 3\frac{(n-1)n}{2} - n \tag{4.70}$$

Now add $3n^2$ to both sides, to get $3S(n) = n^3 - \frac{3}{2}n^2 + \frac{3}{2}n - n + 3n^2$ From which 4.4 follows. To derive 4.4 we must employ the fourth powers in the same way.

Example 4.21 The sum of the first n odd integers is n^2. We see this using 4.4. The first n odd integers are the numbers $1, 3, 5, \ldots, 2n-1$. The kth odd integer is $2k-1$. Thus the sum of the first n odd integers is

$$\sum_1^n (2k-1) = 2\left(\sum_1^n k\right) - \sum_1^n 1 = (n-1)n - n = n^2\,. \tag{4.71}$$

Sometimes to find the sum of a collection of numbers it helps to write out the first few terms.

Example 4.22 Find $\sum_3^{15}(k^{-1} - (k+1)^{-1})$.

The sum of the first two terms is $(1/3 - 1/4) - (1/4 - 1/5) = 1/3 - 1/5$ Then the sum of the first three terms is $(1/3 - 1/5) + (1/5 - 1/6) = 1/3 - 1/6$. Notice the cancellation. This will happen at each stage, because each time the first term added is the same as the last subtracted. We conclude

$$\sum_3^{15}(k^{-1} - (k+1)^{-1}) = 1/3 - 1/16\,. \tag{4.72}$$

Now let us return to the definition of the definite integral. Let $y = f(x)$ be a continuous function on the interval $[a,b]$. Select $n+1$ points between a and b: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. This is a *partition P* of the interval $[a,b]$. The *size* of the partition is the maximum difference between consecutive points, denoted $|P|$. For the sum (called *the Riemann sum* of the function over the partition)

$$\sum_P f(x)\Delta x = \sum_1^n f(x_k')(x_k - x_{k-1}) \tag{4.73}$$

where x_k' is any point in the interval between x_{k-1} and x_k.

Definition 4.3 *The function f is integrable over the interval $[a,b]$ if the Riemann sums converge. That is, there is a number L for which the following condition can be verified. Given any $\varepsilon > 0$, there is a $\delta > 0$ such that if the partition satisfies $|P| < \delta$, then $\left|\sum_P f(x)\Delta x - L\right| < \varepsilon$. L is the integral, denoted $\int_a^b f(x)dx$.*

Now, we have already observed that all continuous functions are integrable, and we have discovered that the value of the integral is found by evaluating an indefinite integral. However, historically, the above definition (for area) was formulated, in a somewhat vaguer sense, long before the Calculus. And areas were calculated by actually finding this limit. In the sixteenth century Cavalieri succeeded in doing this for the functions $y = x^p$ over the interval $[0,1]$ for all values of p from 1 to 9. This was a huge effort; completely replaced by one-line calculations using the calculus. Here is how Cavalieri proceeded.

Take the particular partition

$$P: \quad 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < \frac{n}{n} = 1\ . \tag{4.74}$$

Here $x_k = k/n$. Form the Riemann sum with x_k' taken to be x_k:

$$\sum_P x^p \Delta x = \sum_1^n \left(\frac{k}{n}\right)^p \left(\frac{k}{n} - \frac{(k-1)}{n}\right) = \sum_1^n \left(\frac{k}{n}\right)^p \left(\frac{1}{n}\right) = \frac{1}{n^{p+1}} \sum_1^n k^p \tag{4.75}$$

Now, for $p = 1$, we obtain

$$\sum_P x\Delta x = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n} \tag{4.76}$$

which converges to $1/2$ as $n \to \infty$. Thus the limit exists, and we can conclude that

$$\int_0^1 x dx = \frac{1}{2}\ . \tag{4.77}$$

For $p = 2$ we obtain

$$\sum_P x^2 \Delta x = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6n^3} \tag{4.78}$$

which converges to 1/3. Thus, taking the limit, we have

$$\int_0^1 x^2 dx = \frac{1}{3}\ . \tag{4.79}$$