Calculus II
Practice Problems 1: Answers

1. Solve for x:

a) $6^x = 36^{2-x}$

Answer. Since $36 = 6^2$, the equation becomes $6^x = 6^{2(2-x)}$, so we must have $x = 2(2-x)$ which has the solution $x = 4/3$.

b) $\ln_3 x = 5$

Answer. If we exponentiate both sides we get $x = 3^5 = 243$.

c) $\ln_2(x+1) - \ln_2(x-1) = \ln_2 8$

Answer. Since the difference of logarithms is the logarithm of the quotient, we rewrite this as

$$\ln_2(\frac{x+1}{x-1}) = \ln_2 8 \ ,$$

which is, after exponentiating, the same as

$$\frac{x+1}{x-1} = 8 \ ,$$

which gives us $x+1 = 8(x-1)$, so that $x = 9/7$.

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2. Find the derivative of the given function:

a) $y = \ln(\ln x)$

Answer. Use the chain rule:

$$\frac{dy}{dx} = \frac{1}{\ln x}\frac{d}{dx}\ln x = \frac{1}{x\ln x}$$

b) $y = \log_2(x^2+1)$

Answer. Remember that $\log_2 A = \ln A/\ln 2$, so $y = (\ln(x^2+1))/\ln 2$. Then, use the chain rule:

$$\frac{dy}{dx} = \frac{1}{(\ln 2)(x^2+1)}2x = \frac{2}{\ln 2}\frac{x}{x^2+1} \ .$$

c) $y = \dfrac{e^{x^2}}{x}$

Answer. Use the quotient rule carefully:

$$\frac{dy}{dx} = \frac{x(2xe^{x^2}) - e^{x^2}}{x^2} = 2e^{x^2} - x^{-2}e^{x^2} \ .$$

$$= e^{x^2}(2 - x^{-2})$$

3. Solve: $\sqrt{\ln x} = \ln(\sqrt{x})$.

Answer. By the laws of exponents, this becomes $\sqrt{\ln x} = (1/2)\ln x$. Squaring both sides, we get the equation $4\ln x = (\ln x)^2$. Thus $\ln x = 0$, or $\ln x = 4$, giving the solutions $x = 1, e^4$.

4. Find the integrals:

a) $$\int \frac{(\ln x)^2 + 1}{x} dx \; =$$

Answer. Let $u = \ln x$, so $du = dx/x$. Then

$$\int \frac{(\ln x)^2 + 1}{x} dx \; = \int (u^2+1)du = \frac{u^3}{3} + u + C = \frac{(\ln x)^3}{3} + \ln x + C \; .$$

b) $$\int e^{\sin x} \cos x dx \; =$$

Answer. Let $u = \sin x, \; du = \cos x dx$. Then

$$\int e^{\sin x} \cos x dx \; = \int e^u du = e^{\sin x} + C \; .$$

c) $$\int \frac{x dx}{3x^2+1} \; =$$

Answer. Let $u = 3x^2 + 1, \; du = 6x dx$. Then

$$\int \frac{x dx}{3x^2+1} \; = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln(3x^2+1) + C \; .$$

5. Solve the initial value problem $(x+1)y' = 2y, \; y(1) = 1$.

Answer. Separating variables, this becomes

$$\frac{dy}{y} = \frac{2dx}{x+1} \; .$$

Integrating both sides,

$$\ln y = 2\ln(x+1) + C \; ,$$

which exponentiates to $y = K(x+1)^2$, where $K = e^C$. The initial values give $1 = K(1+1)^2$, so $K = 1/4$, and the solution is $y = (x+1)^2/4$.

6. If $f(x) = 2\sqrt{x}\ln x$, find $f'(x)$.

Answer. It is always a good idea to switch to exponential notation. Write $f(x) = 2x^{1/2}\ln x$. By the product rule,

$$f'(x) = 2\frac{1}{2}x^{-1/2}\ln x + 2x^{1/2}/x = x^{-1/2}(\ln x + 2)\ .$$

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7. I invest \$100,000 in a company for five years, with a guaranteed income of 8% per year, compounded semi-annually. How much will I have at the end of 5 years? If the interest were compounded continuously, how much would I have in 5 years?

Answer. For the first question, I accrue interest at the rate of 4% per period, for 10 periods. Thus, the amount I have at the end is

$$P = 10^5(1+.04)^{10} = 10^5(1.4802) = 148,020\ .$$

If the interest is compounded continuously, the amount is

$$P = 10^5 e^{.08(5)} = 10^5 e^{.4} = 10^5(1.4918) = 149,180\ .$$

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8. A certain element decays at a rate of .000163/year. Of a piece of this element of 450 kg, how much will remain in ten years?

Answer. At the end of t years, we have $450e^{-.000163t}$ remaining. Thus, the amount after 10 years is $A = 450e^{-.00163} = 449.93$ kg.

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9. Two variables are related by the equation $2\ln x + \ln y = x - y$. What is the equation of the tangent line to the graph of this relation at the point (1,1)?

Answer. Differentiate the equation implicitly:

$$\frac{2}{x} + \frac{y'}{y} = 1 - y'\ .$$

Substituting the values $x = 1$, $y = 1$ gives the slope of the tangent line: $2 + y' = 1 - y'$, so $y' = -1/2$. Then the tangent line is

$$\frac{y-1}{x-1} = -\frac{1}{2}$$

or $2y + x = 3$.

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10. If the region in the first quadrant bounded by the curve $y = e^x$ and $x = 1$ is rotated about the x axis, what is the volume of the resulting solid?

Answer. The region being rotated is that under the curve $y = e^x$ between $x = 0$ and $x = 1$. Now $dV = \pi r^2 dx = \pi e^{2x} dx$, so the volume is

$$\int_0^1 e^{2x} dx = \frac{e^{2x}}{2}\Big|_0^1 = \frac{e^2 - 1}{2}\ .$$