**Calculus II**
**Practice Problems 2: Answers**

1. Solve the initial value problem:

$$4y' + 3y = e^x , \qquad y(0) = 7 .$$

**Answer**. First solve the homogeneous equation, which can be written as $dy/y = -(3/4)dx$, which has the solution $y = Ke^{-(3/4)x}$. We try $y = ue^{-(3/4)x}$ in the original equation. The left hand side is

$$4y' + 3y = 4(ue^{-(3/4)x})' + 3ue^{-(3/4)x} = 4u'e^{-(3/4)x} - 3ue^{-(3/4)x} + 3ue^{-(3/4)x} = 4u'e^{-(3/4)x} ,$$

so the original equation, in terms of $u$ is

$$4u'e^{-(3/4)x} = e^x \qquad \text{or} \qquad u' = \frac{1}{4}e^{(7/4)x} ,$$

which has the solution

$$u = \frac{1}{7}e^{(7/4)x} + C \qquad \text{so that} \qquad y = (\frac{1}{7}e^{(7/4)x} + C)e^{-(3/4)x} = \frac{1}{7}e^x + Ce^{-(3/4)x} .$$

The initial condition gives $C = 48/7$, and the solution is $y = (1/7)(e^x + 48e^{-(3/4)x})$.

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2. Solve the initial value problem:

$$xy' - 3y = x^2 , \qquad y(1) = 4 .$$

**Answer**. First solve the homogeneous equation: $xy' - 3y = 0$:

$$\frac{dy}{y} = 3\frac{dx}{x} \qquad \text{so that} \qquad \ln y = 3\ln x + C ,$$

which gives us $y = Kx^3$. Now try $y = ux^3$ and solve for $u$. The left hand side of the original equation is

$$xy' - 3y = (ux^3)' - 3ux^3 = x(u'x^3 + 3ux^2) - 3ux^3 = x^4u' .$$

So we have to solve $x^4u' = x^2$, or $u' = x^{-2}$, so

$$u = -\frac{1}{x} + C \qquad \text{so that} \qquad y = (-\frac{1}{x} + C)x^3 = Cx^3 - x^2 .$$

The initial values give $4 = C - 1$, so $C = 5$, and our solution is

$$y = 5x^3 - x^2 .$$

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3. If I invest \$ 8,000 at 12.5 percent per year (compounded continuously) in how many years will my investment be worth \$ 30,000 ?

**Answer**. The amount I have after $t$ years is given by $P(t) = P_0e^{rt}$ where $P_0 = 8000$ and $r = .125$. Now for my problem, I want to find $t$ such that $P(t) = 30000$. So, we must solve

$$30000 = 8000e^{(.125)t}$$

for $t$. We get $.125t = \ln(30/8) = 1.3218$, so $t = 1.3218/.125 = 10.57$ years.

4. At what rate (continuously compounded) should I invest \$10,000 so as to have \$14,000 in five years?

**Answer**. Here we have the same equation: $P(t) = P_0 e^{rt}$, but what is given is $P_0 = 10000$, $t = 5$, $P(5) = 14000$, and we are asked to find $r$. We solve

$$14000 = 10000e^{5r} ,$$

obtaining $5r = \ln 1.4$, or $r = .3365/5 = .0673$, so the rate should be 6.73%.

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5. The half-life of $\text{Rossidium}_{312}$ is 4,801 years. How long will it take for a mass of $\text{Rossidium}_{312}$ to decay to 98 % of its original size?

**Answer**. This is once more a growth/decay problem, so the relevant equation is $P(t) = P_0 e^{rt}$. We haven't been given the rate $r$, but we are told the half life. So first, we use that information to find $r$, by solving the basic equation with $P_0 = 1$, $t = 4801$, $P(t) = .5$ : $.5 = e^{4801r}$ , so $r = \ln(.5)/4801 = -1.44 \times 10^{-4}$. Now, we want to solve for $t$, with $P(t) = .98$:

$$.98 = e^{-1.44 \times 10^{-4} t} ,$$

giving

$$t = \frac{\ln .98}{-1.44 \times 10^{-4}} = 140 \text{ years} .$$

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6. According to Newton's Law of Cooling, if a hot object is immersed into a cool environment, the rate of decrease of the temperature of the object is proportional to the difference in the temperature of the object and its environment. If, then, $h(t)$ is the temperature of the object at time $t$, and $T_0$ is the temperature of the environment, Newton's law says

$$\frac{dh}{dt} = -k(h(t) - T_0) , \qquad (1)$$

where $k$ is the coefficient of cooling. Suppose that a body at 95° Celsius is immersed in a water bath held at 5° Celsius, and the coefficient of cooling is $k = .08$. What will be the temperature of the body in 10 minutes?

**Answer**. Given the above information, we have to solve the initial value problem

$$\frac{dh}{dt} = -.08(h - 5), \qquad h(0) = 95 ,$$

and find the value when $t = 10$. First we separate variables:

$$\frac{dh}{h-5} = -.08dt$$

and integrate both sides:

$$\ln(h - 5) = -.08t + C$$

and then exponentiate: $h = 5 + Ke^{-.08t}$. The initial condition tells us what $K$ is: $95 = 5 + K$, so $K = 90$, and the equation is $h = 5 + 90e^{-.08t}$. At $t = 10$, we obtain $h = 5 + 40.44 = 45.44°$.

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7. Suppose that I wish to make iced tea of tea at the boiling point, to be consumed in three minutes. To get the tea as cold as possible, should I put in an ice cube immediately, or just before the three minutes are up?

**Answer**. According to equation (1) in problem 6, the rate of decrease of temperature is proportional to the difference of the temperature of the object and the temperature of the environment, so we can expect largest rate of cooling when the tea is hottest. This suggests that we should cool the tea down first, and then add the ice cube.

To illustrate this, let's compare the two processes, assuming that the effect of adding an ice cube is to drop the temperature by 20%, and that the coefficient of cooling is $k = .3$. We assume the tea is at the boiling point (100°), and the room is at 15°. Following the argument for problem 6, the solution to the differential equation (1) is

$$h(t) = T_0 + (T_i - T_0)e^{-kt} , \tag{2}$$

where $T_i$ is the initial temperature of the hot object.

Now, in the first case, we put in the ice cube first, dropping the temperature of the tea to 80°. Then (2) takes over, so

$$h(3) = 15 + (80 - 15)e^{-.3(3)} = 41.42^\circ .$$

In the second case, natural cooling leads to the temperature

$$h(3) = 15 + (100 - 15)e^{-.3(3)} = 49.56^\circ ,$$

which drops by 20% by adding the ice cube, to a temperature of 39.64°. Not a big difference, but enough to substantiate the conclusion.

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8. A lake containing 300,000 acre-feet of water has 20% salinity. Clear water flows in from rivers, and out at a dam, both at the rate of 4,000 acre-feet per day. In how may days will the salinity be reduced to 10%?

**Answer**. Let $S(t)$ be the amount of salt in the lake at time $t$. Initially we have $S(0) = .2(300) = 60$ thousand acre-feet of salt. On day $t$, the fraction of salt in the lake is $S(t)/300$, so the amount of salt eliminated in that day is $\Delta S = (S/300)(4)$ thousand acre-feet. This gives us the differential equation for the amount of salt in the lake:

$$dS = -\frac{4}{300}Sdt = -.0133Sdt .$$

The solution is $S(t) = Ke^{-.0133t}$, and the initial condition gives $K = 60$. When the lake has 10% salinity, we have $S(t) = 30$. Thus the time at which we have 10% salinity is the solution of

$$30 = 60e^{-.0133t}$$

so that $t = \ln(1/2)/(-.0133) = 52.12$ days.

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9. A pond is in the form of a cylinder of radius 100 ft. and depth 8 ft. Water flows into the pond at the rate of 100 cu. ft./hr and seeps into the ground through the porous bottom at a rate proportional to the volume, where the constant of proportionality is .0005. What is the maximum height of water in the pond that can be achieved? If the water level in the pond is 2 feet at time t= 0, what is the height after 1000 days?

**Answer**. Let $W(t)$ be the amount of water in the pond at time $t$, measured in cubic feet. Taking into account both the inflow and the seepage, we have the differential equation

$$\frac{dW}{dt} = 100 - .0005W .$$

Now, since the problem is about the height of the water in the pond, we change to the variable $h(t)$, the height. Using $V = \pi r^2 h$, the formula for the volume of a cylinder, and $r = 100$, we have $W = 10^4 \pi h$. This gives the differential equation for $h$:

$$10^4 \pi \frac{dh}{dt} = 100 - .0005(10^4 \pi)h = 100 - 5\pi h \ . \tag{3}$$

The maximum height is attained when $h' - 0$, or $100 - 5\pi h = 0$, so is $h = 6.37$ feet. Now, for the second question, we have to solve (3). Separating variables, we get

$$\frac{dh}{5\pi h - 100} = -\frac{1}{10^4 \pi} dt \ ,$$

leading to the solution

$$\frac{1}{5\pi} \ln(5\pi h - 100) = -\frac{t}{10^4 \pi} + C$$

which we can rewrite as

$$\ln(5\pi h - 100) = -.0005t + C \ .$$

Exponentiating, we get

$$h = \frac{100}{5\pi} + Ke^{-.0005t} = 6.37 + Ke^{-.0005t} \ .$$

The initial condition $h = 2$ when $t = 0$, finally gives

$$h = 6.37 - 4.37e^{-.0005t} \ .$$

Now substitute $t = 1000$, and obtain $h = 6.37 - 4.37e^{-.5} = 3.719$ feet.

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10. Water flows into an elastic ball at a rate of 4 cu. in/minute. The ball has a puncture out of which water flows at a rate proportional to the volume of water in the ball, where the constant of proportionality is .02. Assuming the ball is empty at the beginning, how much water is in the ball after 20 minutes?

**Answer**. Letting $W(t)$ be the amount of water in the ball, the data given tells us that the rate of change of $W$ is given by the differential equation

$$\frac{dW}{dt} = 4 - .02W \ .$$

This can be rewritten as $dW/(.02W - 4) = -dt$ which integrates to

$$\frac{1}{.02} \ln(.02w - 4) = -t + C \ ,$$

which integrates to $.02W - 4 = Ke^{-.02t}$. Since the ball is empty at the beginning, $W = 0$ when $t = 0$, we find $K = -4$, and finally we have

$$W = \frac{4}{.02}(1 - e^{-.02t}) \ .$$

Now at $t = 20$, we obtain

$$W = 200(1 - e^{-.4}) = 65.93 \text{ cu. in} \ .$$