Calculus II
Practice Problems 3: Answers

1. Differentiate:

$$f(x) = \sqrt{\frac{2x-6}{3x+5}}\ .$$

Answer. This problem is here to suggest a different way, called *logarithmic differentiation*, of differentiating expressions like this. First we rewrite the function exponentially:

$$f(x) = (2x-6)^{1/2}(3x+5)^{-1/2} \qquad (1)$$

and then we take the logarithm:

$$\ln f(x) = \frac{1}{2}[(\ln(2x-6) - \ln(3x+5)]\ .$$

Now differentiate this equation:

$$\frac{f'(x)}{f(x)} = \frac{1}{2}[\frac{2}{2x-6} - \frac{3}{3x+5}]$$

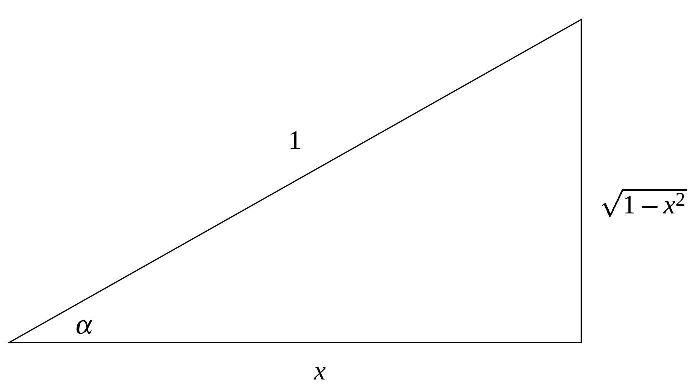
and now multiply through by $f(x)$, as in equation (1):

$$f'(x) = f(x)(\frac{1}{2}[\frac{2}{2x-6} - \frac{3}{3x+5}]) = (2x-6)^{-1/2}(3x+5)^{-1/2} - \frac{3}{2}(2x-6)^{1/2}(3x+5)^{-3/2}$$

2. *a)* $\tan(\arccos x) =$

b) $\frac{1}{x^2} - \tan^2(\arccos x) =$

Answer. a. Draw a right triangle with hypotenuse 1, an angle α, and label the side adjacent to α as x so that $\alpha = \arccos x$ (see the figure).



Now, by the Pythagorean theorem, the side opposite α has length $1-x^2$. Thus

$$\tan(\arccos x) = \tan\alpha = \frac{\sqrt{1-x^2}}{x}$$

Now, for b):

$$\frac{1}{x^2} - \tan^2(\arccos x) = \frac{1}{x^2} - (\frac{\sqrt{1-x^2}}{x})^2 = \frac{1-(1-x^2)}{x^2} = 1 \ .$$

From the diagram, $1/x = \sec\alpha$, so this also follows from the identity $\tan^2\alpha + 1 = \sec^2\alpha$.

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3. Differentiate $y = \arccos\sqrt{x}$

Answer. By the chain rule:

$$y' = -\frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x(1-x)}}$$

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4. Differentiate: $g(x) = \arcsin(\ln x)$.

Answer. By the chain rule:

$$g'(x) = \frac{1}{x\sqrt{1-(\ln x)^2}}$$

5. Integrate

a) $\displaystyle\int_0^2 \frac{xdx}{1+4x^2} =$

b) $\displaystyle\int_0^2 \frac{dx}{1+4x^2} =$

Answer. For a), let $u = 1+4x^2,\ du = 8xdx$. Then

$$\int_0^2 \frac{xdx}{1+4x^2} = \frac{1}{8}\int_1^{17}\frac{du}{u} = \frac{1}{8}\ln u\Big|_1^{17} = \frac{\ln 17}{8}$$

For b), however, we should make the substitution: $u = 2x,\ du = 2dx$. Then

$$\int_0^2 \frac{dx}{1+4x^2} = \frac{1}{2}\int_0^4 \frac{du}{1+u^2} = \frac{1}{2}\arctan u\Big|_0^4 = \frac{\arctan 4}{2}$$

This problem illustrates that one must be careful in deciding what substitution to make. One has to scan the integrand to see how to write it as a product of a function of u (the substitution to be made) and the differential of u.

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6. Integrate:

a) $\displaystyle\int \frac{e^x dx}{e^{2x}+1} =$

b) $\displaystyle\int \frac{dx}{e^x + e^{-x}} =$

Answer. For part a) recognize that for $u = e^x,\ du = e^x dx$, and thus

$$\int \frac{e^x dx}{e^{2x}+1} = \int \frac{du}{1+u^2} = \arctan u + C = \arctan e^x + C \ .$$

For part b), we see that, in order to make the same substitution, we need an e^x in the numerator. So put it there, by multiplying by $e^x/e^x = 1$:

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^x}\frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \arctan e^x + C \ ,$$

by part a).

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7. $$\int \frac{dx}{\sqrt{5-4x-x^2}} =$$

Answer. To get this to look like something recognizable, we complete the square: $5-4x-x^2 = 9-(x+2)^2$. The problem now looks like

$$\int \frac{dx}{\sqrt{9-(x+2)^2}} = \ .$$

This looks like the integral should involve arccos, but we need a 1 where the 9 is. We fix that by the substitution $3u = x+2$, $3du = dx$. Then we have

$$\int \frac{dx}{\sqrt{9-(x+2)^2}} = \int \frac{3du}{\sqrt{9-9u^2}} = \int \frac{du}{\sqrt{1-u^2}} = -\arccos u + C = -\arccos\frac{x+2}{3} + C \ .$$

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8. Integrate:

a) $$\int \tan^2 x dx =$$

Answer. $\tan^2 x = \sec^2 x - 1$, so

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C \ .$$

b) $$\int \tan^3 x dx =$$

Answer. Using the same identity

$$\int \tan^3 x dx = \int (\sec^2 x - 1)\tan x dx = \int \sec x(\sec x \tan x) dx - \int \tan x dx$$

$$= \frac{\sec^2 x}{2} + \ln(\cos x) + C \ ,$$

using the substitution $u = \sec x,\ du = \sec x \tan x dx$ for the first integral.

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9. $\int x\tan(x^2+1)dx =$

Answer. Let $u = x^2+1,\ du = 2xdx$. Then

$$\int x\tan(x^2+1)dx = \frac{1}{2}\int \tan u du = -\frac{1}{2}\ln(\cos u) + C = -\frac{1}{2}\ln(\cos(x^2+1)) + C \ .$$

10. $$\int \frac{dx}{x^2 - 6x + 13} =$$

By completing the square, we see that $x^2 - 6x + 13 = (x-3)^2 + 4$. Let $u = x - 3,\ du = dx$, so that we have

$$\int \frac{dx}{x^2 - 6x + 13} = \int \frac{dx}{(x-3)^2 + 4} = \int \frac{du}{u^2 + 4} \ .$$

Now, we know the integral of $du/(u^2+1)$; but how do we do $du/(u^2+4)$? Again a little algebra comes into play:

$$u^2 + 4 = 4((\frac{u}{2})^2 + 1) \ .$$

Try the substitution $v = u/2,\ dv = du/2$:

$$\int \frac{du}{u^2+4} = \frac{1}{4}\int \frac{2dv}{v^2+1} = \frac{1}{2}\arctan v + C = \frac{1}{2}\arctan\frac{x-3}{2} + C \ .$$