Calculus II
Practice Problems 5: Answers

1. $\lim_{x\to 0} \frac{\sin x - x}{x(\cos x - 1)} =$

Answer. l'Hôpital's rule will apply, and continues to apply as we use it, so we get the train of equalities

$$\lim_{x\to 0} \frac{\sin x - x}{x(\cos x - 1)} =^{l'H} \lim_{x\to 0} \frac{\cos x - 1}{\cos x - 1 - x\sin x} =^{l'H} \lim_{x\to 0} \frac{-\sin x}{-2\sin x - x\cos x} =^{l'H}$$

$$\lim_{x\to 0} \frac{-\cos x}{-3\cos x + x\sin x} = \frac{1}{3} ,$$

since the last limit can be evaluated at the limit point.

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2. $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} =$

Answer. Again, l'Hôpital's rule will apply through this train of equalities:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} =^{l'H} \lim_{x\to 0} \frac{e^x - 1}{2x} =^{l'H} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2} .$$

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3. $\lim_{x\to 1} \frac{\ln x}{\cos((\pi/2)x)} =$

Answer. After verifying that the hypotheses of l'Hôpital's rule hold:

$$\lim_{x\to 1} \frac{\ln x}{\cos((\pi/2)x)} =^{l'H} \lim_{x\to 1} \frac{1/x}{-(\pi/2)\sin((\pi/2)x)} = \frac{1}{-\pi/2} = -\frac{2}{\pi} .$$

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4. $\lim_{x\to 0^+} (\frac{\cos(\sqrt{x}) - 1}{x}) =$

Answer.

$$\lim_{x\to 0^+} (\frac{\cos(\sqrt{x}) - 1}{x}) =^{l'H} \lim_{x\to 0^+} \frac{-\sin(\sqrt{x})\frac{1}{2\sqrt{x}}}{1} = -\lim_{x\to 0^+} \frac{\sin(\sqrt{x})}{2\sqrt{x}} = -\frac{1}{2}$$

since $\sin u/u \to 1$ as $u \to 0$.

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5. $\lim_{x\to 5}(\frac{5\cos(\pi x) + x}{x^2 - 25}) =$

Answer. We can use the same methods since $\cos(5\pi) = -1$:

$$\lim_{x\to 5}(\frac{5cos(\pi x) + x}{x^2 - 25}) =^{l'H} \lim_{x\to 5} \frac{-5\pi\sin(\pi x) + 1}{2x} = \frac{1}{10} ,$$

since $\sin(5\pi) = 0$.

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6. $\lim_{x\to 1^+}(x-1)\ln(\ln x) =$

Answer. Since $x-1\to 0$ and $\ln(\ln x)\to -\infty$, this is of the form $0\cdot\infty$, and we have to invert one of the factors. We find

$$\lim_{x\to 1^+}(x-1)\ln(\ln x) = \lim_{x\to 1^+}\frac{\ln(\ln x)}{1/(x-1)} =^{l'H} \lim_{x\to 1^+}\frac{\frac{1}{x\ln x}}{-(x-1)^{-2}} = -\lim_{x\to 1^+}\frac{(x-1)^2}{x\ln x}\ ,$$

using a little algebra. Now, apply l'Hôpital's rule again:

$$=^{l'H} \lim_{x\to 1^+}\frac{2(x-1)}{\ln x+1} = 0\ .$$

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7. $\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} =$

Answer. l'Hôpital's rule applies, and we get

$$\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} =^{l'H} \lim_{x\to\infty}\frac{1}{x/\sqrt{1+x^2}} = \lim_{x\to\infty}\frac{\sqrt{1+x^2}}{x}\ .$$

Thus the number we are looking for is its own inverse, so is ± 1. However, since the values, as $x\to+\infty$ are positive, the limit must be 1.
We note that a little algebra will do as well:

$$\lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{\frac{1}{x^2}+1}} = 1\ .$$

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8. $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) =$

Answer. This is of the form $\infty-\infty$, so we need to start by using some algebra. Putting the expression over a common denominator, we find

$$\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) = \frac{x-1-\ln x}{(x-1)\ln x}\ .$$

Now we can use l'Hôpital's ule:

$$\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x} =^{l'H} \lim_{x\to 1}\frac{1-1/x}{\frac{x-1}{x}+\ln x} = \lim_{x\to 1}\frac{x-1}{x-1+x\ln x} =^{l'H} \lim_{x\to 1}\frac{1}{1+\ln x+1} = \frac{1}{2}\ .$$