Calculus II
Practice Problems 1: Answers

Find the limits.

1. $\lim_{n\to\infty} \frac{n}{(\ln n)^{15}}$

Answer. Here the general term is of the form $f(n)$, where $f(x) = x/(\ln x)^{15}$. We can then switch to the function f and use l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{x}{(\ln x)^{15}} =^{l'H} \lim_{x\to\infty} \frac{1}{15(\ln x)^{14}(1/x)} = \lim_{x\to\infty} \frac{x}{15(\ln x)^{14}}$$
$$=^{l'H} \lim_{x\to\infty} \frac{x}{(15)(14)(\ln x)^{13}} = \cdots$$

We can see where this is going: after 13 more steps, we get

$$\lim_{x\to\infty} \frac{x}{(\ln x)^{15}} = \frac{1}{15!} \lim_{x\to\infty} \frac{x}{\ln x} =^{l'H} \frac{1}{15!} \lim_{x\to\infty} \frac{1}{1/x} = \infty \; .$$

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2. $\lim_{n\to\infty} \frac{n^k}{n!}$

Answer. Well, this is just example 5 in section 9.1.

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3. $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2$

Answer. $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty} (1+\frac{1}{n})^2 = 1$.

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4. $\lim_{n\to\infty} \frac{(2n-1)^2}{n^2-3n+1}$

Answer. $\lim_{n\to\infty} \frac{(2n-1)^2}{n^2-3n+1} = \lim_{n\to\infty} \frac{4n^2-4n+1}{n^2-3n+1} = 4$.

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5. $\lim_{n\to\infty} \frac{(1+n)^n}{n!}$

Answer. Both the numerator and denominator of the nth term have n factors; but each factor of the numerator is larger than the corresponding factor of the denominator. Thus the nth term is greater than the quotient of the last two factors:

$$\frac{(1+n)^n}{n!} = \left[\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n-1}\right)\left(\frac{n+1}{n-2}\right)\cdots\right]\frac{n+1}{1} \geq \frac{n+1}{1} \; .$$

Thus the sequence diverges to ∞.

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6. $\lim_{n\to\infty} n^{1/n}$

Answer. We can calculate this by l'Hôpital's rule, replacing n by x. First we take logarithms: $\ln(x^{1/x}) = \ln x/x$. Now

$$\lim_{x\to\infty} \frac{\ln x}{x} =^{l'H} \lim_{x\to\infty} \frac{1/x}{1} = 0 \; .$$

Thus $\lim_{n\to\infty} n^{1/n} = \exp(\lim_{n\to\infty} \ln x/x) = e^0 = 1$.

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7. $\sum_{n=1}^{\infty} \frac{5^n}{8^{n+1}}$

Answer. $\sum_{n=1}^{\infty} \frac{5^n}{8^{n+1}} = \frac{1}{8}\big(\sum_{n=1}^{\infty} \frac{5^n}{8^n}\big) = \frac{1}{8}\big(\sum_{n=0}^{\infty} \frac{5^n}{8^n} - 1\big) = \frac{1}{8}\big(\frac{1}{1-\frac{5}{8}} - 1\big) = \frac{1}{3} - \frac{1}{8}$

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8. $\sum_{n=1}^{\infty} \frac{5^n}{8^n+1}$

Answer. $\sum_{n=1}^{\infty} \frac{5^n}{8^n+1} < \sum_{n=1}^{\infty} \frac{5^n}{8^n} < \infty$.

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9. $\sum_{k=1}^{\infty} \frac{1}{(2k)(2k+2)}$

Answer. Suspecting a telescoping series, we find the partial fraction expansion:

$$\frac{1}{(2k)(2k+2)} = \frac{1}{2}\big(\frac{1}{2k} - \frac{1}{2k+2}\big) \ ,$$

so

$$\sum_{k=1}^{n} \frac{1}{(2k)(2k+2)} = \frac{1}{2}\big(\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \cdots + \frac{1}{2n} - \frac{1}{2n+2}\big) = \frac{1}{2}\big(\frac{1}{2} - \frac{1}{2n+2}\big) \ ,$$

which converges to 1/4.

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10. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Answer. This is just example 11 of section 9.2. However, it is interesting to note that we can actually sum this series, using a clever trick due to Oresme (14th century. The formula for the sum of the geometric series is classical and appears in Euclid).

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} \big(\sum_{k=n}^{\infty} \frac{1}{2^k}\big) = \sum_{n=1}^{\infty} \big(\frac{1}{2^{n-1}}\big) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \ ,$$

using the results of examples 7 and 8.