Calculus II
Practice Problems 9: Answers

In problems 1-5 find the radius of convergence of the series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)!} x^n$

Answer. We use the ratio test:

$$\frac{2^{n+1}}{(n+2)!}\frac{(n+1)!}{2^n} = \frac{2}{n+2} \to 0.$$

Thus the radius of convergence is $R = \infty$.

---

2. $\sum_{n=1}^{\infty} \frac{n}{3^n} x^n$

Answer. Again, the ratio test:

$$\frac{n+1}{3^{n+1}}\frac{3^n}{n} = (\frac{n+1}{n})\frac{1}{3} \to \frac{1}{3} ,$$

so $R = 3$.

---

3. $\sum_{n=0}^{\infty} n(n-1)(n-2)(\frac{x}{3})^n$

Answer. We look at the ratios:

$$\frac{(n+1)n(n-1)}{3^{n+1}}\frac{3^n}{n(n-1)(n-2)} = (\frac{n+1}{n-2})\frac{1}{3} \to \frac{1}{3} ,$$

so $R = 3$.

---

4. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$

Answer. We look at the ratios:

$$\frac{(2n+2)!}{((n+1)!)^2}\frac{(n!)^2}{(2n)!} = \frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{(2+\frac{2}{n})((2+\frac{1}{n})}{(1+\frac{1}{n})^2} \to 4 .$$

Thus $R = 1/4$.

---

5. $\sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)}{n!} x^n$

Answer. Here we observe that the general term is of the form $a_n = (p(n)/n!)x^n$, where p is a polynomial of degree 3, Thus $a_n = (g(n))(x^n/(n-3)!)$, where g is a ratio of polynomials of degree 3. Since $\lim g(n)$

exists, the numbers $g(n)$ are bounded. Thus, by the comparison test (with the exponential series), this series converges for all x; that is, $R = \infty$.

---

6. Let $f(x) = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{n!} x^n$. Find a formula for the function f.

Answer. We have to look at the form of the series: is it obtained from a series we know using algebraic operations, differentiation, or integration? The terms $n+2$, $n+1$ give a clue: these have been obtained by two differentiations. So, we integrate twice to see what we have:

$$F(x) = \int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{n+2}{n!} x^{n+1} ,$$

$$G(x) = \int_0^x F(t)dt = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+2} ,$$

Now, we can factor out a x^2, and obtain

$$G(x) = x^2 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = x^2 e^x .$$

Then

$$F(x) = G'(x) = 2xe^x + x^2 e^x , \quad f(x) = F'(x) = 2e^x + 4xe^x + x^2 e^x .$$

Here's another one to try:

$$f(x) = \sum_{n=0}^{\infty} (n+2)(n+1) x^n .$$

---

7. Find the Taylor series centered at the origin for the function $F(x) = \int_0^x \frac{dt}{1-t^4}$.

Answer. We start with the series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n .$$

Substitute t^4 for x, and then integrate term by term:

$$\frac{1}{1-t^4} = \sum_{n=0}^{\infty} t^{4n} ,$$

$$F(x) = \int_0^x \frac{dt}{1-t^4} = \sum_{n=0}^{\infty} \frac{1}{4n+1} t^{4n+1} .$$

---

8. Find the Taylor series centered at the origin for the antiderivative (indefinite integral) of $f(x) = \frac{e^{-x^2} - 1}{x}$.

Answer. We start with the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

Now, substitute $-x^2$ for x and subtract 1:

$$e^{-x^2} - 1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{n!} .$$

Now, divide by x:

$$\frac{e^{-x^2}-1}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{n!} \ .$$

Now integrate term by term:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)n!} x^{2n} \ .$$

---

9. Find the Taylor series centered at the origin for the function $\int_0^x \frac{1+t^2}{1-t^2} dt$.

Answer. We start with the series

$$\frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n} \ .$$

Then

$$\frac{t^2}{1-t^2} = \sum_{n=0}^{\infty} t^{2n+2} \ ,$$

so that

$$(1+t^2)(\frac{1}{1-t^2}) = \sum_{n=0}^{\infty} t^{2n} + \sum_{n=0}^{\infty} t^{2n+2} \ .$$

Now, the second series in this sum is the same as the first, except for the first term. Thus

$$\frac{1+t^2}{1-t^2} = 1 + 2\sum_{n=1}^{\infty} t^{2n} \ .$$

Now integrate term by term:

$$\int_0^x \frac{1+t^2}{1-t^2} dt = x + 2\sum_{n=1}^{\infty} \frac{1}{2n+1} x^{2n+1} \ .$$

---

10. Find the Taylor series centered at the origin for the function $\frac{1}{(1-x^2)^2}$.

Answer. We start with the series

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \ .$$

Differentiate both sides;

$$\frac{2x}{(1-x^2)^2} = \sum_{n=1}^{\infty} 2nx^{2n-1} \ .$$

Now divide both sides by $2x$, and change the index:

$$\frac{1}{(1-x^2)^2} = \sum_{n=1}^{\infty} nx^{2(n-1)} = \sum_{n=0}^{\infty} (n+1)x^{2n} \ .$$

We remark that one could also start with the series for $1/(1-x)$, differentiate it, and then substitute x^2 for x, leading to, of course, the same result.

---

11. Find the Taylor expansion of x^3 centered at the point -1.

Answer. We need to find the successive derivatives of $f(x) = x^3$ at $x = -1$. We find: $f(-1) = -1$, $f'(-1) = 3$, $f''(-1) = -6$, $f'''(-1) = 6$, and all subsequence derivatives are zero. Thus the Taylor expansion is

$$x^3 = -1 + 3(x+1) + \frac{-6}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3 = -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3 .$$

---

12. Find the Taylor series centered at the origin for the function $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Answer. We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} , \qquad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} .$$

When we add the two series, the odd terms cancel, and the even terms double. Thus

$$e^x + e^{-x} = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} ,$$

so

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} .$$

---

13. Find the first 5 coefficients of the Maclaurin series for $f(x) = e^x \cos x$.

Answer. We have to write down the Maclaurin series of e^x and $\cos x$ up to the fifth term, and then multiply just as we would polynomials, throwing away products which are of degree greater than 5;

$$e^x \cos x = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots)(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} - \frac{x^5}{12} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{30}x^5 + \cdots$$

---

14. Expand $f(x) = 1 + x - 3x^2 + x^9$ in a Maclaurin series.

Answer. The expression of a polynomial as a sum of monomials *is* its expression as a Maclaurin series.