VI. Transcendental Functions

6.1 Inverse Functions

The functions e^x and $\ln x$ are inverses to each other in the sense that the two statements

$$y = e^x \ , \qquad x = \ln y$$

are equivalent. In general, two functions f, g are said to be *inverse to each other* when the statements

$$y = f(x) \ , \qquad x = g(y) \tag{6.1}$$

are equivalent for x in the domain of f, and y in the domain of g. Often we write $g = f^{-1}$ and $f = g^{-1}$ to express this relation. Another way of giving this citerion is

$$f(g(x)) = x \qquad g(f(x)) = x \ .$$

Example 6.1. Find the inverse function for $f(x) = 3x - 7$. We write $y = 3x - 7$ and solve for x as a function of y:

$$x = \frac{y+7}{3} \ . \tag{6.2}$$

The equations $y = 3x - 7$ and $x = (y + 7)/3$ are equivalent for all x and y, so (6.2) gives us the formula for the inverse of f: $f^{-1}(y) = (y + 7)/3$. Since it is customary to use the variable x for the independent variable, we should write:

$$f^{-1}(x) = \frac{x+7}{3} \ .$$

Example 6.2. Find the inverse function for

$$f(x) = \frac{x}{x+1} \ .$$

We let $y = x/(x + 1)$, and solve for x in terms of y:

$$yx + y = x \qquad \text{so that} \qquad y = x(1 - y) \ , \tag{6.3}$$

so that

$$x = \frac{y}{1-y} \ .$$

Thus

$$f^{-1}(x) = \frac{x}{1-x} \ .$$

Notice that -1 is excluded from the domain of f, and 1 is excluded from the domain of f^{-1}. In fact, we see that these substitutions in equations (6.3) lead to contradictions.

We have to be careful, in discussing inverses, to clearly indicate the domain and range, otherwise we have ambiguities and make mistakes.

Example 6.3. x^2 and $\sqrt{x}$ appear to be inverses since $(\sqrt{x})^2 = x$. But this doesn't work if x is negative, since the symbol $\sqrt{\ }$ just gives the positive root, $\sqrt{x^2} = |x|$. To make the statement accurate, we have to specify the domain of the squaring function: the function $f(x) = x^2$ whose domain is the set of nonnegative numbers, has the inverse function $g(x) = \sqrt{x}$. When the domain is taken to be all numbers, $f(x) = x^2$ does not have an inverse, since, for each positive number y, there are two values of x such that $x^2 = y$. Note, though, if we specify the domain of $f(x) = x^2$ to be all nonpositive numbers, then it has the inverse defined for all nonnegative numbers: $h(x) = -\sqrt{x}$.

We illustrate this graphically in figure 6.1.

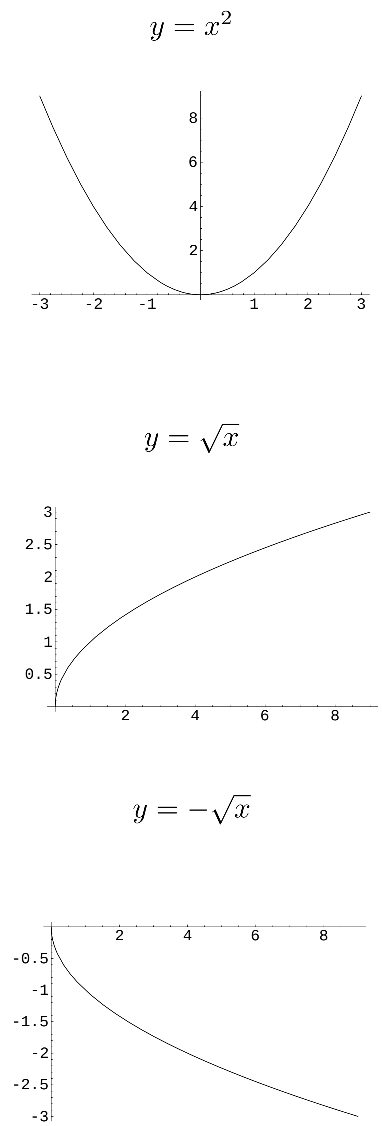


Figure 6.1

In the first graph each horizontal line $y = y_0$ intersects the graph in two points for $y_0 > 0$, and in no points for $y_0 < 0$. So the domain of an inverse function can contain no negative numbers, and for positive numbers, there are 2 choices of inverse, one for the function x^2, x nonnegative, and the other for x^2, x nonpositive.

In general, this provides a graphical criterion for a function to have an inverse:

Proposition 6.1. Let $y = f(x)$ for a function f defined on the interval $a \le x \le b$. Let $f(a) = \alpha$, $f(b) = \beta$. If, for each γ between α and β the line $y = \gamma$ intersects the graph in one and only one point, then f has an inverse defined on the interval between α and β.

For if (c, γ) is the point of intersection of the graph with the line $y = \gamma$, define $f^{-1}(\gamma) = c$.

For a continuous function, we know, from the Intermediate Value Theorem of Chapter 2, that each such line $y = \gamma$ intersects the graph in at least one point. Thus for continuous functions, we can restate the proposition as

Proposition 6.2. Let $y = f(x)$ for a continuous function f defined on the interval $a \le x \le b$. Let $f(a) = \alpha$, $f(b) = \beta$. If the condition

$$x_1 \neq x_2 \qquad \text{implies} \qquad f(x_1) \neq f(x_2) \tag{6.4}$$

then f has an inverse defined on the interval between α and β.

For a differentiable function, it follows from Rolle's theorem of chapter that condition (6.4) holds if $f'(x) \neq 0$ for all $a \le x \le b$.

Proposition 6.3. Let $y = f(x)$ for a differentiable function f defined on the interval $a \le x \le b$. Let $f(a) = \alpha$, $f(b) = \beta$. If $f'(x) \neq 0$ in the interval, then f has an inverse defined on the interval between α and β.

Example 6.4. Let $f(x) = x^2 - x$. Find the domains for which f has an inverse, and find the inverse function.

First, differentiate: $f'(x) = 2x - 1$. Thus $f'(x) < 0$ for $x < 1/2$, and $f'(x) > 0$ for $x > 1/2$, so we should be able to find inverses for f on each of the domains $(-\infty, 1/2), (1/2, \infty)$.To find the formula for the inverse, let $y = x^2 - x$ and solve for x in terms of y. To do this, we write the equation as $x^2 - x - y = 0$, and use the quadratic formula:

$$x = \frac{-1 \pm \sqrt{1+4y}}{2} \ .$$

How convenient: we're looking for two possible inverses, and here we have two choices. Notice first that because of the square root sign, the domain of y must be $y \ge -1/4$. We conclude that, in the domains $x \ge 1/2$, $y \ge -1/4$ the following statements are equivalent:

$$y = x^2 - x \ , \qquad x = \frac{-1 + \sqrt{1+4y}}{2}$$

and thus the inverse to $f(x) = x^2 - x$ defined for $x \ge 1/2$ is the function defined on the domain $x \ge -1/4$ by

$$f^{-1}(x) = (-1 + \sqrt{1+4x})/2 \ . \tag{6.5}$$

Similarly, in the domains $x \leq 1/2,\ y \geq -1/4$ the following statements are equivalent:

$$y = x^2 - x \ , \qquad x = \frac{-1 - \sqrt{1+4y}}{2}$$

and thus the inverse to $f(x) = x^2 - x$ defined for $x \leq 1/2$ is the function defined on the domain $x \geq -1/4$ by by $f^{-1}(x) = (-1 - \sqrt{1-4x})/2$.

Example 6.5. Let

$$f(x) = \frac{e^x - e^{-x}}{2} \ .$$

This function is called the *hyperbolic sine*. The hyperbolic sine has an inverse function defined for all real numbers. First of all $f'(x) = (e^x + e^{-x})/2 > 0$ for all x, so f has an inverse function. Secondly,

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = \infty$$

so the range of f, and thus the domain of its inverse, is all real numbers. We now find a formula for the inverse function. Let $y = f^{-1}(x)$, so that

$$x = f(y) = \frac{e^y - e^{-y}}{2} \ .$$

Multiply both sides of the equation by $2e^x$, giving

$$2xe^y = e^{2y} - 1 \qquad \text{or} \qquad e^{2y} - 2xe^y - 1 = 0 \ .$$

Using the quadratic formula we find

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1} \ .$$

Since this is positive for all x, we must have $e^y = x + \sqrt{x^2+1}$, and finally

$$y = \ln(x + \sqrt{x^2+1})$$

is the inverse hyperbolic sine.

Proposition 6.4. Suppose that f and g are differentiable functions inverse to each other in their respective domains. Let $y = g(x)$. Then

$$g'(x) = 1/f'(y) \ . \tag{6.6}$$

To see this, differentiate the relations $x = f(y),\ y = g(x)$ implicitly with respect to x:

$$1 = f'(y)\frac{dy}{dx} \ , \quad \frac{dy}{dx} = g'(x) \ ,$$

so

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(y)} \ .$$

Example 6.6. Let us illustrate this proposition with the exponential and logarithmic functions. Recall that $y = \ln x$ is defined as being equivalent to $x = e^y$. Differentiate that equation with respect to x implicitly .

$$1 = e^y \frac{dy}{dx} \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{e^y} \ .$$

Since $e^y = x$, we obtain the formula for the derivative of the logarithm:

$$\frac{d}{dx} \ln x = \frac{1}{x} \ .$$

Example 6.7. Let $y = f^{-1}(x)$ be the function defined on the domain $x \geq 2$ which is inverse to $f(x) = x^2 - x$ (recall example 6.4). We find the derivative of $f^{-1}(x)$ .First, write:

$$y = f^{-1}(x) \quad \text{is equivalent to} \quad x = y^2 - y \ .$$

Differentiate implicitly:

$$1 = 2y\frac{dy}{dx} - \frac{dy}{dx} \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{2y-1} \ .$$

or

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{2f^{-1}(x) - 1} \ . \tag{6.7}$$

Since we have an explicit formula for $f^{-1}(x)$ (see equation (6.5)), we may substitute that in (6.7) to obtain

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{\sqrt{1+4x}} \ .$$

Of course, in the above example the inverse functions are explicit, and so we can make a substitution for $f^{-1}(x)$ on the left side of (6.7), but that may not always be the case.

Example 6.8. Suppose that g is the inverse to the function $f(x) = x^2 - 4x - 44$ for $x > 2$. Find $g'(1)$.

Note, since the parabola has its vertex where $x = 2$, the function f does have an inverse in $x > 2$. Let $y = g(x)$. Since g is inverse to f, $x = f(y) = y^2 - 4y - 44$ and $f'(y) = 2y - 4$, so

$$g'(x) = \frac{1}{2y-4} \ .$$

To calculate $g'(1)$ we find the value of y corresponding to $x = 1:\ 1 = y^2 - 4y - 44$ has the solutions $-9, 5$. Since f is restricted to values greater than 2, we must have $g(1) = 5$. Now $f'(y) = 2y - 4$, so

$$g'(1) = \frac{1}{f'(5)} = \frac{1}{2(5)-4} = \frac{1}{6} \ .$$

Problems 6.1

1. Find the function inverse to

$$f(x) = \frac{2x+1}{x-3} \ .$$

2. Consider the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

with domain the set of positive numbers. Show that, on this domain, f has an inverse, and find the inverse function.

3. Find the inverse function, and its domain, for

$$f(x) = \frac{e^x + e^{-x}}{2} \ .$$

If possible, find a formula for f^{-1}. 4. Consider the function

$$f(x) = \frac{x}{x^2 + 1} \ ,$$

with domain the set of all real numbers. Graph the function, and observe that by considering the intervals $(-\infty, -1)$, $(-1, 1)$, $(1, +\infty$ as distinct domains for the function f, that f has an inverse on each of these intervals. Find those inverses and their domains.

5. Consider $f(x) = x + \sqrt{x^2 + 1}$ with domain the set of positive real numbers. Show that f has an inverse, and find the inverse function.

6. Find $g'((e + e^{-1})/2)$ where g is the inverse to the function of problem 2.

7. Show that $f(x) = x^3 + 3x + 1$ has an inverse. Find

$$\frac{d}{dx} f^{-1}(x)\big|_{x=1} \ .$$

8 . Let $f(x) = x \ln x$ for $x > 1$. Show that f has an inverse g. Noting that $f(e^2) = 2e^2$, find $g'(2e^2)$.

6.2 Inverse Trigonometric Functions

In this section we use the ideas of the preceding section to define inverses for the trigonometric functions, and calculate their derivatives. Since the trigonometric functions are periodic, we will have to restrict the domain of definition in order to obtain a well-defined inverse.

We start with the tangent function. Recall that $\tan x$ is strictly increasing on the interval $(\pi/2, \pi/2)$ and takes every value between $-\infty$ and ∞, and then repeats itself in intervals of length π. Thus, if we restrict the domain of the tangent to the interval $(\pi/2, \pi/2)$, it has an inverse there, defined for all real numbers.

Definition 6.1. The function $y = \arctan x$ is defined on the interval $(-\infty, \infty)$, taking values in $(-\pi/2, \pi/2)$ .by the condition $x = \tan y$.

The inverse tangent (or *arctangent*) is sometimes denoted by $y = \tan^{-1}(x)$. See figure 6.2 for the graph of the inverse tangent.

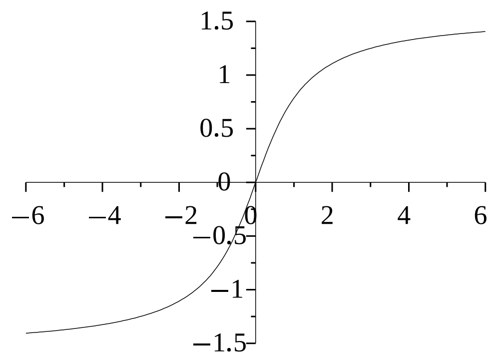


Figure 6.2

Proposition 6.5. $$\frac{d}{dx}\arctan x = \frac{1}{(1+x^2)} \ , \qquad \int \frac{1}{(1+x^2)}dx = \arctan x + C$$

To see this, we start with the equation $x = \tan y$ that defines y as the arctangent of x. We get:

$$1 = \sec^2 y \frac{dy}{dx} \ .$$

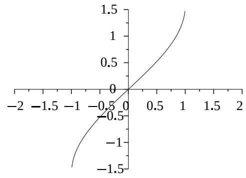
Now, since $\sec^2 y = \tan^2 y + 1$, we can replace $\sec^2 y$ by $x^2 + 1$, obtaining

$$1 = (x^2+1)\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{x^2+1} \ ,$$

which is just the first equation. The second is a restatement in terms of integrals.

Similarly, we define $y = \arcsin x$ by the condition $x = \sin y$. However, since the sine function is periodic, the equation $\sin y = x$ has many solutions for x between -1 and 1. But, if we insist that y be between $-\pi/2$ and $\pi/2$, there is only one solution. So, to pick a definite inverse for the sine function, we specify that its domain is the interval $[-1, 1]$, and its range (set of values) is $[-\pi/2, \pi/2]$. Then, with this specification, it is true that the equation $\sin y = x$ has one and only one solution. That solution we call the *inverse sine function*, denoted $\arcsin x$ or $\sin^{-1} x$.

Definition 6.2. The function $y = \arcsin x$ is defined on the interval $(-1, 1)$, taking values in $-\pi/2, \pi/2]$ .by the condition $x = \sin y$. See figure 6.3 for a graph of $y = \arcsin x$.



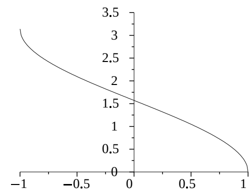


Figure 6.3 Figure 6.4

Proposition 6.6. $$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}\ , \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

Differentiate $x = \sin y$ implicitly:

$$1 = \cos y \frac{dy}{dx}\ .$$

Now, since $\sin^2 y + \cos^2 y = 1$, writing this as $x^2 + \cos^2 y = 1$, and thus replace $\cos y$ by $\sqrt{1-x^2}$:

$$1 = \sqrt{1-x^2}\frac{dy}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\ .$$

We took the positive root for, in the chosen domain for $\arcsin x$, it is increasing.

Turning to the cosine, since $\cos(-x) = \cos(x)$, it is not possible to define an inverse if we take the domain of cos to be any interval about 0. However, we note that since the cosine function is strictly decreasing between 0 and π, we can define an inverse on the interval $(-1, 1)$ taking values between 0 and π: this is the *inverse cosine*, denoted $\arccos x$. (See figure 6.4 for the graph).

Definition 6.3. The function $y = \arccos x$ is defined on the interval $(-1, 1)$, taking values in $(0, \pi)$, by the condition $x = \cos y$.

Proposition 6.7. $$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}\ , \qquad \int \frac{1}{\sqrt{1-x^2}}dx = -\arccos x + C$$

The verification is the same as that of proposition 6.6, except that this time, since the arccosine is decreasing, we take the negative square root. Note that, for any acute angle α, its complementary angle is $\pi/2 - \alpha$, thus $\sin\alpha = \cos(\pi/2 - \alpha)$. Letting $x = \sin\alpha$, so that $\alpha = \arcsin x$, this tells us that $\arccos x = \pi/2 - \alpha = \pi/2 - \arcsin x$, explaining the coincidence in the formulas of propositions 6.6 and 6.7.

Example 6.9. Find

$$\int \frac{xdx}{x^4+1}\ .$$

Make the substitution $u = x^2,\ du = 2xdx$. This gives us

$$\frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x^2) + C\ .$$

Example 6.10. Find, for any constant a:

$$\int \frac{dx}{x^2+a^2}\ .$$

Make the substitution $x = au,\ dx = adu$. The integral becomes

$$\int \frac{adu}{a^2u^2+a^2} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\arctan u + C = \frac{1}{a}\arctan(\frac{u}{a}) + C\ .$$

Problems 6.2

1. $\tan(\arccos x) =$

2. $\frac{1}{x^2} - \tan^2(\arccos x) =$

3. Show that $\arcsin x + \arccos x$ is constant.

4. Differentiate : $g(x) = \arcsin(\ln x)$.

5. Differentiate : $y = \arccos\sqrt{x}$

6. Find the equation of the line tangent to the curve $y = \arctan x$ at the point $(\sqrt{3}, \pi/3)$.

7. Find all points at which the tangent line to the curve $y = \arcsin x$ has slope 4.

8. What is the maximum value of the derivative of $f(x) = \arccos x$?

9. $$\int \frac{xdx}{\sqrt{1-x^4}} =$$

10. Show that $f(x) = \sec x$ has an inverse in the interval $(0, \pi/2)$. The inverse is denoted $y = \sec^{-1} x$ (called the *arcsecant*). Find the formula for the derivative of the arcsecant.

11. $$\int \frac{dx}{\sqrt{a^2-x^2}} =$$

12. The curve $$y = \frac{1}{\sqrt{1+x^2}} \qquad 1 \le x \le \sqrt{3}$$

is rotated around the x-axis. Find the volume of the enclosed solid.

6.3 First Order Linear Differential Equations

Definition 6.4. A *first order linear differential equation* is a differential equation of the type

$$\frac{dy}{dx} + P(x)y = Q(x) \ . \tag{6.8}$$

It is said to be *homogeneous* if the function $Q(x)$ is 0.

The equation is of "first order" since it involves only the first derivative, and linear since the equation expresses the first derivative of the unknown function y as a linear function of y.

If P and Q are constant functions we can easily solve the differential equation by separation of variables.

Example 6.11. To solve, say

$$\frac{dy}{dx} = 2y - 3$$

we rewrite the equation in the form $(2y-3)^{-1}dy = dx$. These differentials integrate to the relation

$$\frac{1}{2}\ln(2y-3) = x + C \quad \text{or} \quad \sqrt{2y-3} = Ke^x \ .$$

Squaring both sides and solving for y, we get the general solution

$$y = \frac{Ke^{2x}+3}{2} \ . \tag{6.9}$$

For example, to find the solution with initial value $y(0) = 5$, we first solve for K:

$$5 = \frac{Ke^{2(0)}+3}{2} \ ,$$

so $K = 7$, and the particular solution is $y = (7e^{2x}+3)/2$.

The acute reader will object that the integral of $(2y-3)^{-1}dy$ is $(1/2)\ln|2y-3|$, and if we follow through with this, this seems to lead to the alternative solution

$$y = \frac{3-Ke^{2x}}{2} \ . \tag{6.10}$$

However, this is the same as (6.9), just with a different choice for the constant K. If we use (6.10) with the same initial conditions $y(0) = 5$, we find this $K = -7$, giving the same final answer. For this reason it is often the case that the absolute value is ignored.

Now, we note that the homogeneous equation (the case $Q(x) = 0$) is separable:

Example 6.12. Solve $y' - 2xy = 0, \ y(2) = 1$.

We separate the variables: $y^{-1}dy = 2xdx$ and integrate:

$$\ln y = x^2 + C \ .$$

Substituting the initial condition allows us to solve for C : $\ln 1 = 4 + C$, so $C = -4$. Thus the particular solution is given by

$$\ln y = x^2 - 4$$

which exponentiates to

$$y = e^{x^2-4} \ .$$

Now, to solve the general equation, we make a crucial observation:

Proposition 6.8. Given the differential equation, $y' + P(x)y = Q(x)$, suppose that v solves the homogeneous equation: $v' + Pv = 0$. Then, making the substitution $y = uv$ leads to a simple integration for the unknown function u.

Let's make the substitution in the given equation. Since $y' = uv' + u'v$, we have

$$uv' + u'v + Puv = Q \ , \qquad \text{or} \qquad u'v + u(v' + Pv) = Q \ , \qquad \text{or} \qquad u'v = Q \ ,$$

since $v' + Pv = 0$. But then $u' = Qv^{-1}$, and we find u by integration.

This leads to a method for solving the general first order differential equation

$$y' + Py = Q \ .$$

1. Find a solution v of the corresponding homogeneous equation.
2. Make the substitution $y = uv$, leading to an integration to find the new unknown function u.

Example 6.13. *Solve* $\quad \dfrac{dy}{dx} = \dfrac{y+1}{x} \ , \quad y(1) = 2 \ .$

The homogeneous equation is $y' - x^{-1}y = 0$. which has the solution $y = Kx$. Try $y = ux$ in the given equation. This leads us to the equation $u'x = x^{-1}$, or $u' = x^{-2}$, which has the solution $u = -x^{-1} + C$. Thus the general solution is

$$y = ux = (\frac{-1}{x} + C)x = -1 + Cx \ .$$

Now solve for C using the initial conditions $y(1) = 2$: $2 = -1 + C$, so $C = 3$ and the solution is $y = 3x - 1$.

Now the solution of the homogeneous equation $y' + Py = 0$ is $e^{-\int P dx}$. With the substitution $y = ue^{-\int P dx}$, the terms involving an undifferentiated u disappear precisely because $e^{-\int P dx}$ solves the homogeneous equation. For this reason $e^{-\int P dx}$ is called an *integrating factor*. This method is called that of *variation of parameters*; the idea being to first find the general solution of an easier equation, and then trying that in the original equation, but with the constant replaced by a new unknown function. This method is very productive in solving very general types of differential equations.

Example 6.14. Solve $y' - 2xy = x$, $y(0) = 2$. First, as in example 6.12, solve the homogeneous equation $y' - 2xy = 0$, leading to

$$y = Ke^{x^2} \ .$$

Now substitute $y = ue^{x^2}$ into the original equation to obtain

$$u'e^{x^2} = x \quad \text{or} \quad u' = xe^{-x^2} \ .$$

This integrates to

$$u = -\frac{1}{2}e^{-x^2} + C \ ,$$

so that our general solution is $y = ue^{x^2}$ with this u:

$$y = \big(-\frac{1}{2}e^{-x^2} + C\big)e^{x^2} = -\frac{1}{2} + Ce^{x^2} \ .$$

Notice that the constant function $-1/2$ (found by taking $C = 0$) is a solution of the differential equation. However, this doesn't satisfy our initial condition: $y(0) = 2$. This gives us $C = 5/2$, so the solution we seek is

$$y = -\frac{1}{2} + \frac{5}{2}e^{x^2} \ .$$

Example 6.15. Find the general solution to $xy' - y = x^2$.

We first must put this in the form (6.8):

$$\frac{dy}{dx} + \frac{y}{x} = x \ .$$

The solution to the homogeneous equation is $y = Kx$. So, we try $y = ux$, and obtain the equation

$$u'x = x \ ,$$

which has the general solution $u = x + C$. Thus the general solution to the original problem is

$$y = ux = (x + C)x = x^2 + Cx \ .$$

Remember the steps to solve the equation $y' + P(x)y = Q(x)$:

1. Solve the homogeneous equation $y' + P(x)y = 0$, obtaining $y = e^{-\int P dx}$.

2. Try the solution $y = ue^{-\int P dx}$, leading to the equation for u : $u'e^{-\int P dx} = Q(x)$, or $u' = Q(x)e^{\int P dx}$.

Solve for u, and put that solution in the equation $y = ue^{-\int P dx}$. If an initial value is specified, now solve for the unknown constant.

This can, of course, be summarized in a formula:

Proposition 6.9 The general solution of the first order linear differential equation

$$y' + Py = Q$$

is

$$y = e^{-\int P dx}\left(\int Q e^{\int P dx} dx + C \right) .$$

We strongly advise students to remember the method rather than this formula.

A useful fact to know about linear first order equations is that if we know one particular solution, then we only have to solve the homogeneous equation to find all solutions.

Proposition 6.10. Suppose that y_p is a solution of the differential equation $y' + Py = Q$. Then every solution is of the form

$$y = y_p + Ke^{-\int P dx} ;$$

that is, every solution is of the form $y_p + y_h$, where y_h is a solution of the homogeneous equation.

For suppose that y is any solution of the equation: $y' + Py = Q$. Then $(y - y_p)' + P(y - y_p) = (y + Py) - (y_p + Py_p) = Q - Q = 0$ so solves the homogeneous equation.

Example 6.16. Find the solution of the equation $y' - 2y + 5 = 0$ such that $y(0) = 1$.

Now the constant function $y_p = 5/2$ solves the equation, since $y_p' = 0$. The general solution of the homogeneous equation is $y = Ke^{2x}$, so the general solution of the original equation is of the form $y = (5/2) + Ke^{2x}$. Substituting $y = 1,\ x = 0$, we find $1 = 5/2 + K$, so $K = -3/2$, and the particular solution we want is

$$y = \frac{1}{2}(5 - 3e^{2x}) .$$

Example 6.17. A body falling through a fluid is subject to the force due to gravity as well as a resistance, due to the viscosity of the fluid, proportional to its velocity. (Here we are assuming that the density of the body is much higher than the density of the fluid, and that its shape is not relevant). Let $x(t)$ represent the distance fallen at time t and $v(t)$ its velocity. The hypothesis leads to the equation

$$\frac{dv}{dt} = -kv + g$$

for some constant k (g is the acceleration of gravity), called the coefficient of resistance of the fluid. Notice that the constant $v = g/k$ is a solution of the equation. This is called the "free fall velocity", and for any falling body it will accelerate until it reaches this maximum velocity. By proposition 6.10, the general solution is

$$v(t) = \frac{g}{k} + Ke^{-kt} ,$$

for some constant k.

Example 6.18. Suppose a heavy spherical object is thrown from an airplane at 10000 meters, and that the coefficient of resistance of air is $k = 0.02$. Find the velocity as a function of time. What is the free fall velocity? Approximately how long does it take to reach the ground?

Here $g = 9.8$ m/sec^2, so the free fall velocity is $v_p = 9.8/(.05) = 196$ meters/sec. The general solution to the problem is

$$v(t) = 196 + Ke^{-(.02)t} .$$

At $t = 0$, $v = 0$, so $0 = 196 + K$, and our solution is

$$v(t) = 196(1 - e^{-(.02)t}) \ .$$

To answer the last question, we have to find distance fallen as a function of time, by integrating the above:

$$x(t) = 196(t + 50e^{-(.02)t}) + C \ .$$

At $t = 0$, $x = 0$; this gives $C = -196(50)$, and the solution for our particular object:

$$x(t) = 196(t + 50(e^{-(.02)t} - 1)) \ .$$

Now we want to solve for t when $x = 10,000$. For large t, the exponential term is negligible, so T, the time to reach ground, is approximately given by the solution of

$$10,000 = 196(T - 50)$$

so $T = 101$ seconds.

Problems 6.3

1. Solve the initial value problem $xy' + y = x,\ \ y(2) = 5$.

2. Solve the initial value problem: $y' = x(5 - y),\ \ \ y(0) = 1$.

3. Solve the initial value problem $(x + 1)y' = 2y,\ y(1) = 1$.

4. Solve the initial value problem $xy' - y = x^3,\ \ y(1) = 2$.

5. Solve the initial value problem $y' - 2xy = e^{x^2},\ y(0) = 4$.

6. Solve the initial value problem:

$$4y' + 3y = e^x \ , \qquad y(0) = 7 \ .$$

7. Solve the initial value problem:

$$xy' - 3y = x^2 \ , \qquad y(1) = 4 \ .$$

8. Solve the initial value problem $y' - 2xy = e^{x^2},\ y(0) = 4$.

9. Solve the initial value problem: $y' + y = e^x,\ \ y(0) = 5$.

10. Solve the initial value problem : $\qquad y' + \dfrac{y}{x} = x, \quad y(1) = 2$.