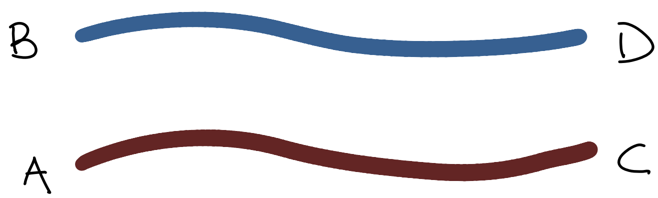
Tangles and fractions

1. The basic operations

To successfully perform today's activities we'll need some rope, 2 pieces of about 10 feet in length, as well as couple of plastic bags, and 4 volunteers. Our initial position is:



Front of the classroom (you)

There are two basic operations: Twist and Rotate. To twist, student A walks under the rope that student B is holding. This is the only twisting move that is allowed. There is no "untwist" move (that would undo the twist). See Figure 2 to see the result of 1, and 2 twists (starting from the neutral position). To rotate, volunteers all rotate one position clockwise viewed from above, as in Figure 3.

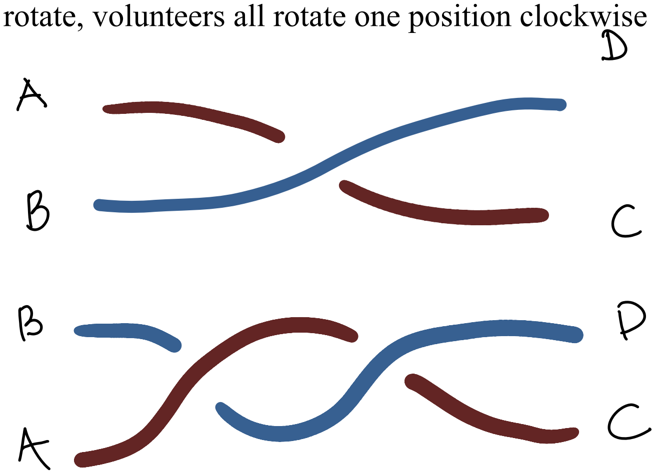


Figure 2: Twisting

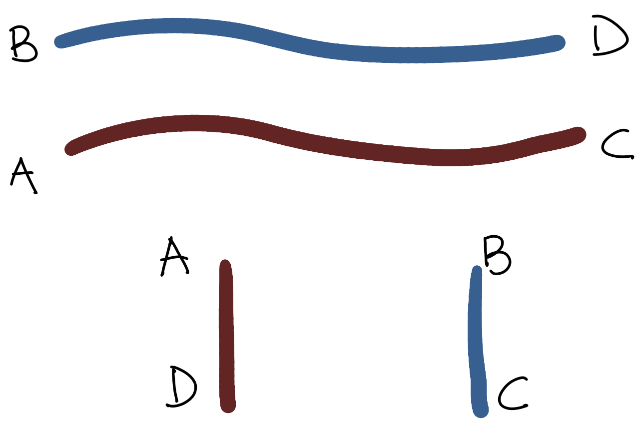


Figure 3: Rotating

We do not actually care about what position the people are in. What we care about is the position of the ropes. For instance, how are the initial position, the position after 2 rotate and the position after 4 rotate related?

In describing a sequence of moves, we will write “T” for twist and “R” for rotate. We will write a sequence of moves by writing something like TRTT to mean start in initial position, then twist, twist, rotate and then twist, in that order.

2. Tangles and numbers

One goal is to associate a number to each tangle. Here are a couple “rules” to get us started to determine how to do this.

- The starting position is given the number 0.
- Each time a twist is done, the number increases by 1 (so the number is an attempt to measure the number of twists made): $t(x) = x + 1$

Question 1: What function describes R?

Experiment 1: How do T and RT compare?

Experiment 2: Start at 0 and perform a twist, rotate, twist.

Experiment 2: Consider RTT, and TRTT. How are they related?

Experiment 3: What might TR be?

3. How do you get back to zero?

Here our goal is to start with a tangle and get it back to the 0-tangle.

Experiment 1: Start at 0 and do three twists. Can you get back to 0?

Experiment 2: Make the TRT^3RT^2.

So, what is the procedure and is it guaranteed to work? In other words, will the procedure that you have outlined bring every fraction down to 0?

4. GCD: Greatest Common Divisor

There is the Euclidean algorithm for computing GCD: if m and n are given two numbers and $m > n$, we can write $m = kn + l$, where $k \geq 1, |l| < n$. Any number that divides m and n, must also divide l, so GCD(m, n) = GCD(n, l). Note that the Euclidean algorithm still works if we use negative numbers in our calculations. In other words, if n is smaller than m, then find the first k for which $kn > m$, then subtract l. Here it is for GCD(5, 17):

5	=	$17 \times 1 - 12$				
17	=	$12 \times 1 + 5$	=	$12 \times 2 - 7$		
12	=	$7 \times 1 + 5$	=	$7 \times 2 - 2$		
7	=	$2 \times 1 + 5$	=	$2 \times 2 + 3$	$= 2 \times 4 - 1$	
2	=	$1 \times 1 + 1$	=	$1 \times 2 + 0$		

How is that related to this sequence?

$$-\frac{5}{17} \xrightarrow{T} \frac{12}{17} \xrightarrow{R} -\frac{17}{12} \xrightarrow{TT} \frac{7}{12} \xrightarrow{R} -\frac{12}{7} \xrightarrow{TT} \frac{2}{7} \xrightarrow{R} -\frac{7}{2} \xrightarrow{TTTT} \frac{1}{2} \xrightarrow{R} -\frac{2}{1} \xrightarrow{TT} 0$$

What is going on? Why are these two operations so similar?

5. What tangle numbers are possible?

Let's start with easy fractions. Can you start with 0 and get to −3?

Can you get from 0 to any (positive or negative) fraction?

6. Can we give any meaning to the idea of an irrational tangle?

What fractions are generated if we generate tangles by a repetition of the sequence RT^3?

References

[1] Davis, Tom Conway's Rational Tangles, http://www.geometer.org/mathcircles/, 2007
[2] Pearson, Mike: Tangles, http://nrich.maths.org/content/id/5681/Tangles.pdf