Potato Theory
(Following Bangert, Franks, Hingston and J. Danforth Quayle)

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[27-Feb-2001]

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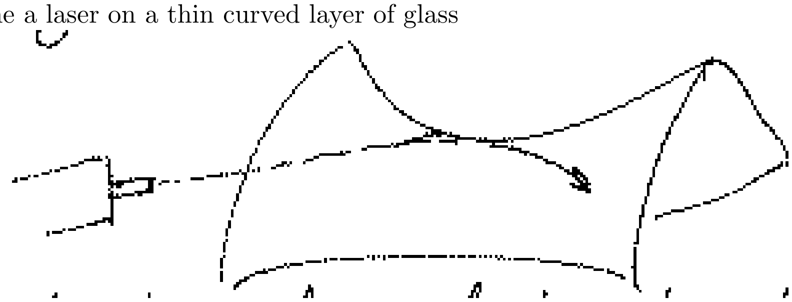
Shine a laser on a thin curved layer of glass



It gets trapped and travels along the glass.

We imagine it travels on one of

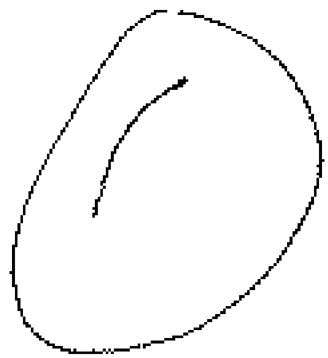
(a) the shortest curve between any two points it hits

(b) the straightest cure between any to points it hits

(c) the curve requiring its lazy photons to expend the least energy.

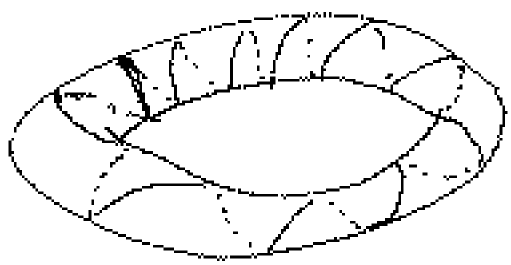
All three ideas describe the same curves: geodesic.

You get the same curves if you pull a string tight, at least on a convex surface with the string on the outside.

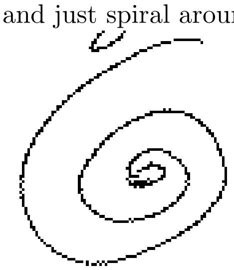


Does the laser light make the whole surface glow, or just strip near the geodesic?

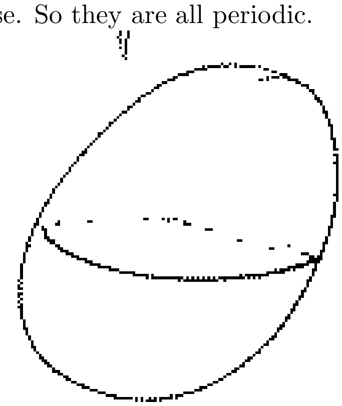
It depends on whether the geodesic winds all over the surface or not.



It might wind around and come back to where it started, or it might wind all over the surface, never returning and "filling up" the whole surface, or it might get stuck in the same region and just spiral around.

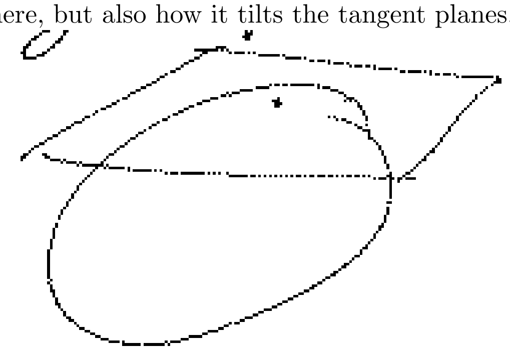


On a perfect sphere, you get equatorial circles (or longitude lines, etc.) as geodesics, and only these. So they are all periodic.

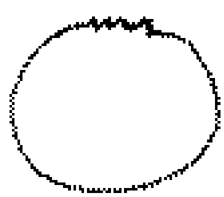


What if it isn't perfect? A slight, hard to notice, imperfection will keep geodesics close to those of a perfect sphere "for a long time," and gradually the laser light will diffuse, so we won't see the difference.

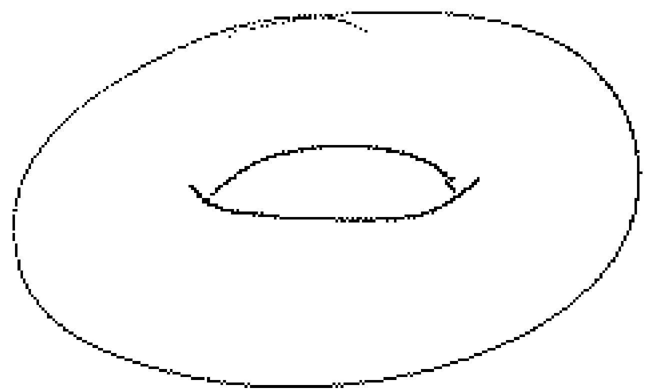
This depends on the imperfection being small not only in how far it deforms points of the sphere, but also how it tilts the tangent planes.



No little bumps



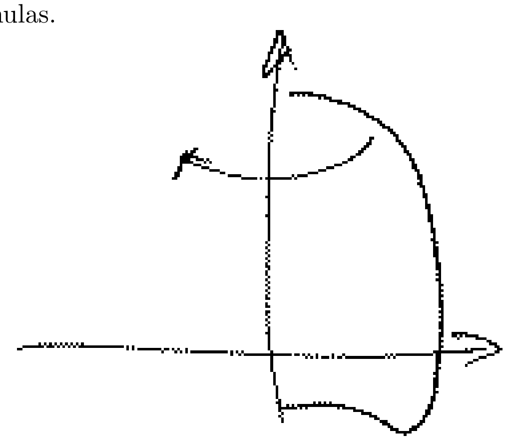
On a torus of revolution



there are periodic geodesics, but you will never find them with a laser, because they are very "rare". A laser will actually light up the whole surface.

Can we find surfaces other than the sphere whose geodesics are all periodic?

Answer: (Zoll) Yes. In fact, even surfaces of revolution, given by explicit but complicated formulas.

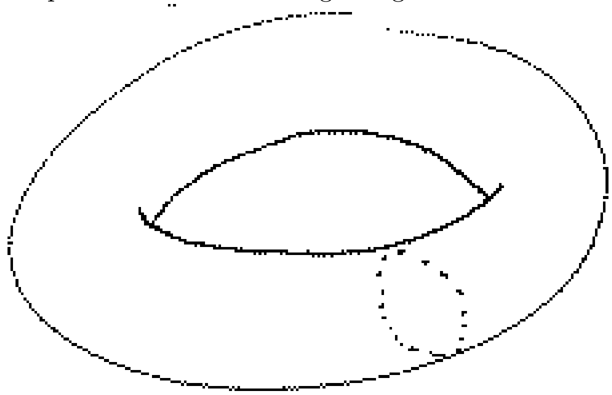


Revolve this curve, and get a Zoll surface.

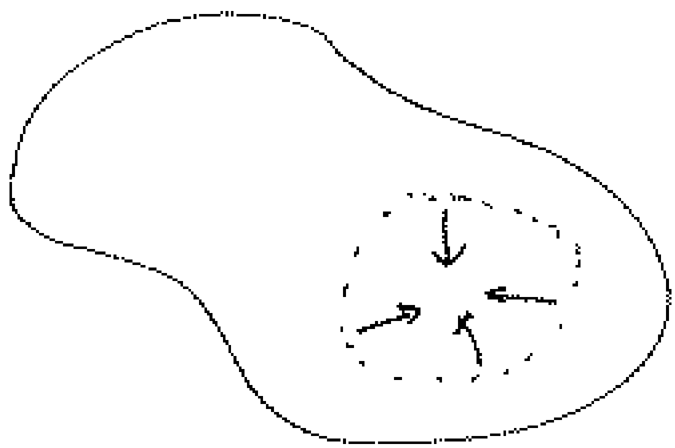
Open problem: can any two Zoll surfaces be deformed into one another through a family of Zoll surfaces?

How can we prove that a surface has a periodic geodesic?

Trick: take a loop and shrink it until it gets tight



Problem: If you have a potato (no doughnut holes) then any loop just shrinks down to a point.



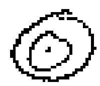
A potato means a smooth deformed sphere.

No doughnut holes, no edges, finite extent.

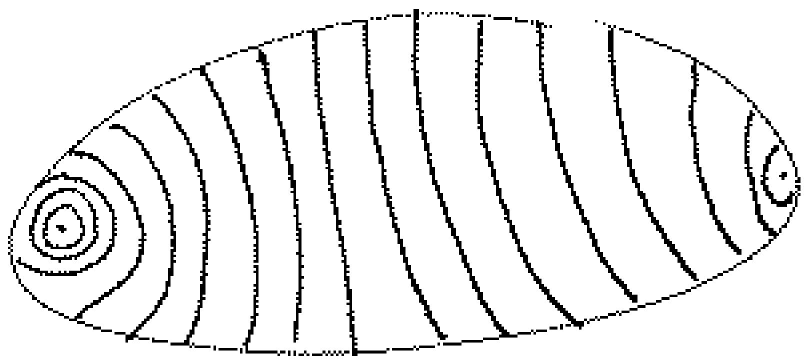
Do potatoes have periodic geodesics?

Idea of Birkhoff (1916): imagine a loop starting off lie at a point,

then expanding out

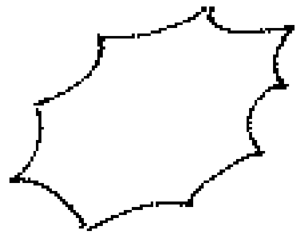


and then contracted again "at the other side".

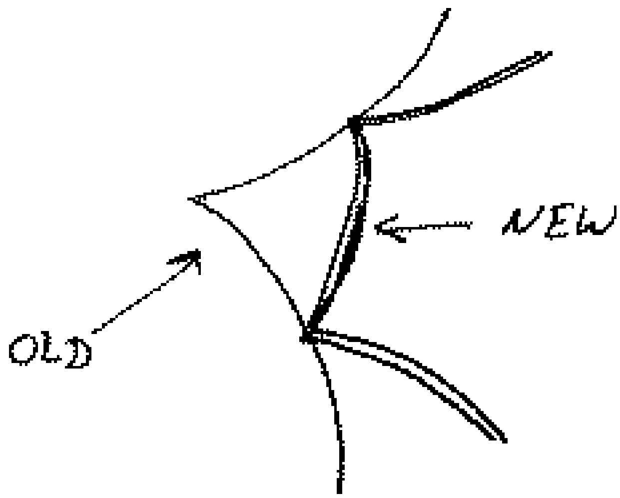


Call this picture a tapestry.

Consider only those tapestries where loops are made entirely of broken geodesics.



Now we can take any loop like this and shrink it:

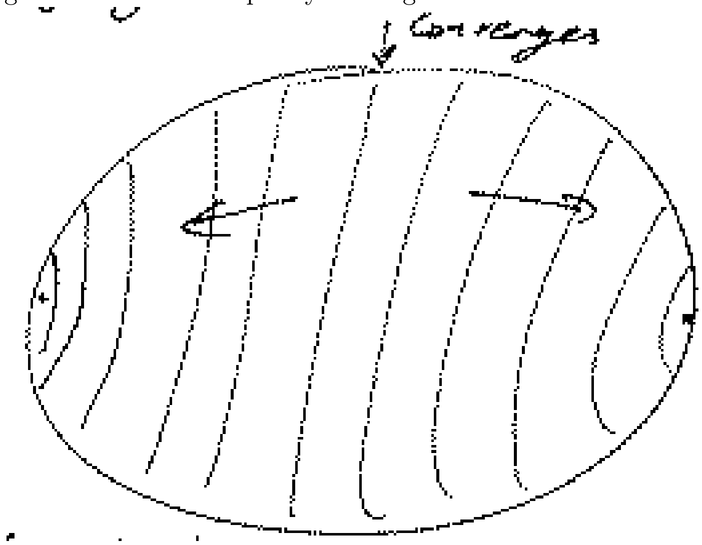


Replace the old break points by the midpoints of the old edges.

Then connect these points by geodesic arcs

With some details, we can prove a triangle inequality so that the new loop is shorter than the old one, unless the old one was already a periodic geodesic.

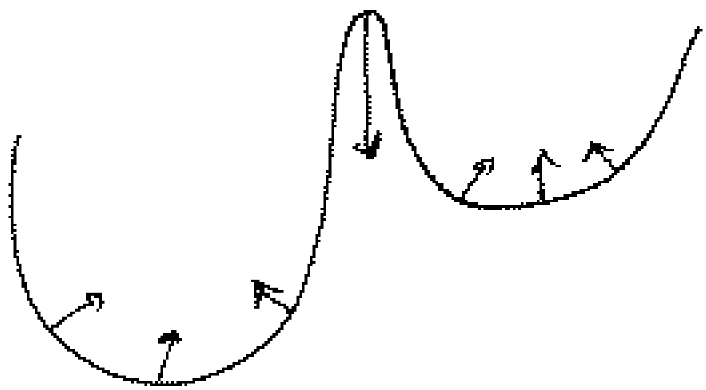
Applying this to an entire tapestry converges



made out of broken geodesics, we find that one loop in the tapestry converges to a periodic geodesic, while the others will usually get pulled away down to points.

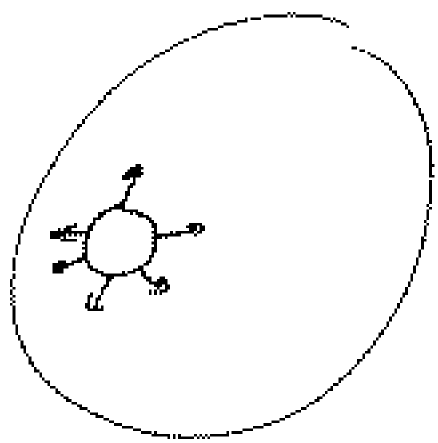
Another approach (Gage & Hamilton 1986) Geodesics are straightest paths.

Measure how curved a loop is (its curvature k) and then push it a little so as to try to get less curvature:



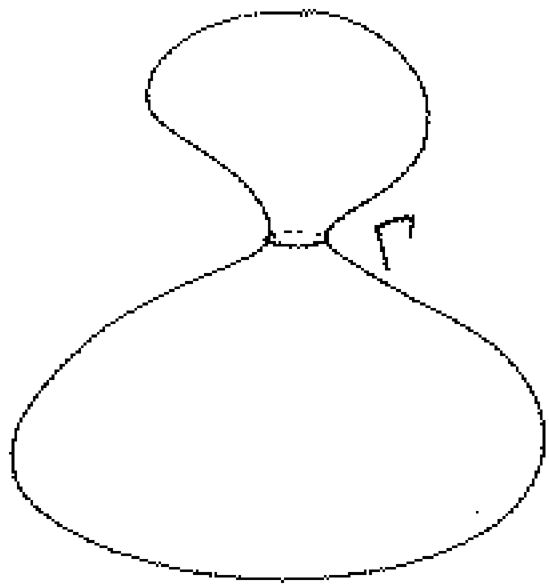
A velocity field "moves" the curve; velocity is large where curvature is large.

This method, applied cleverly, actually gives 3 periodic geodesics on any potato.

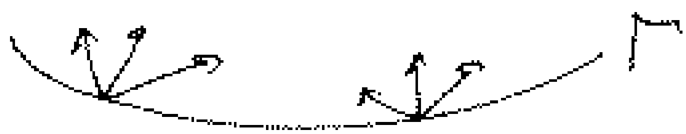


It pulls away curve until it becomes a geodesic.

How that we have one (in fact three) periodic geodesics, we use this to construct more

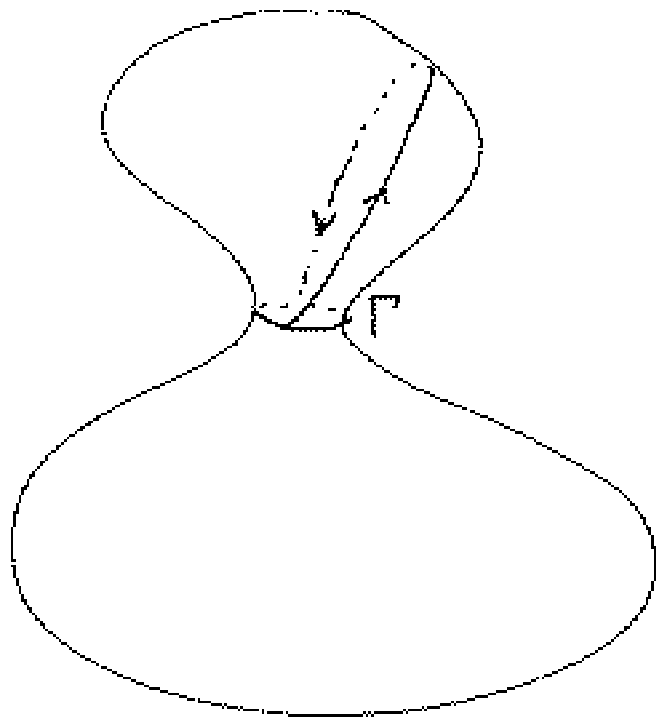


Take a periodic geodesic and shoot geodesics across it

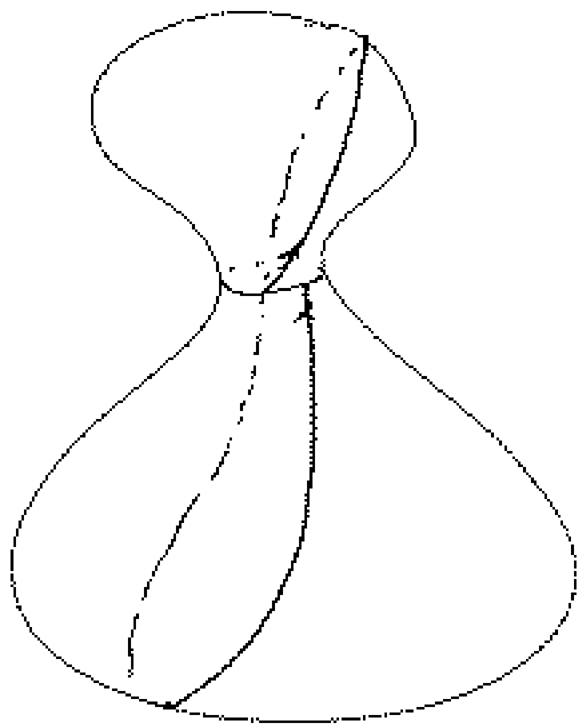


in all possible directions that point toward a given side of Γ.

Watch each one go as it passes up, and then comes back through Γ:

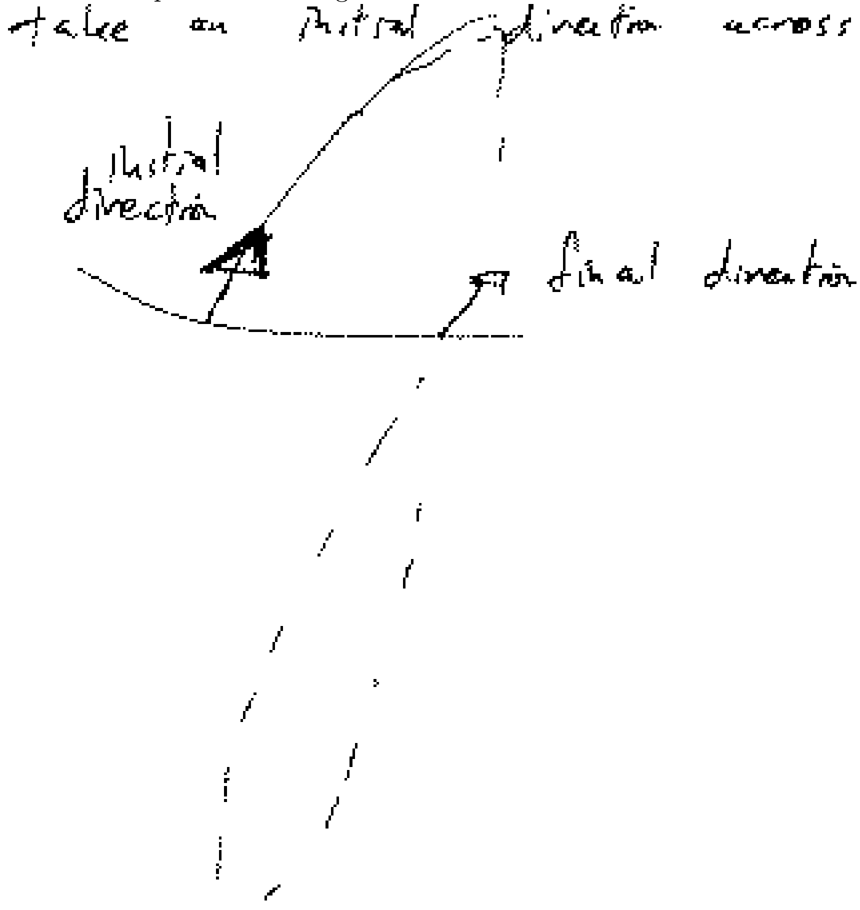


Now let it keep going until it hits Γ again.



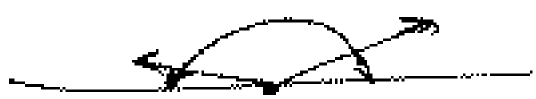
It comes back, in general somewhere else.

Think of this picture as telling us how to take an initial direction across Γ.

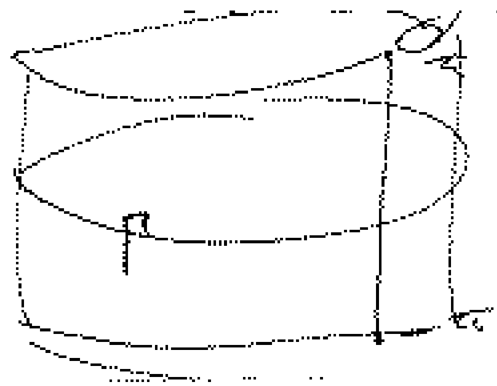


to a final direction.

Directions here are specified by knowing what point of Γ you are at, and an angle between $-\pi$ and π.

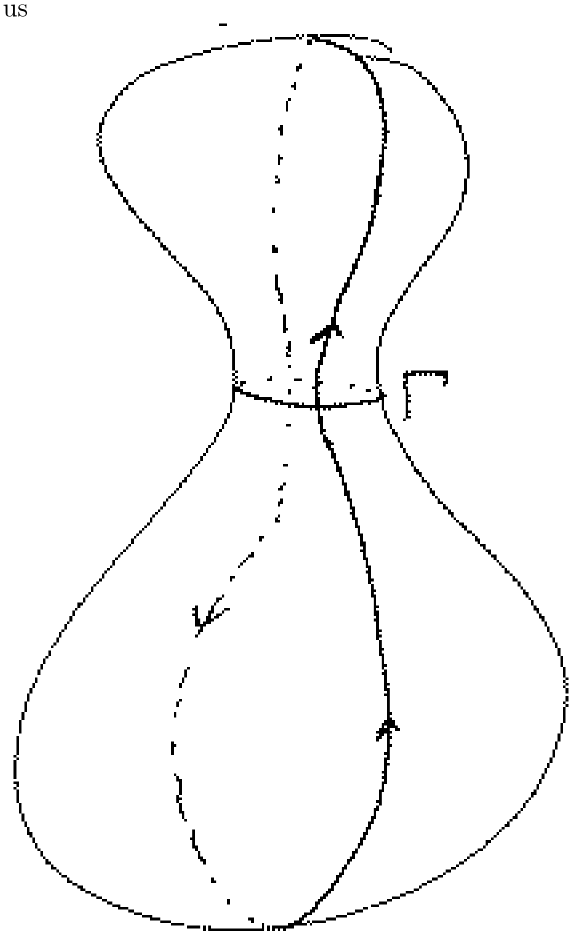


We thus have a cylinder



Theorem 1 [Poincaré-Brinkhoff 1927]
An area preserving map of a cylinder to itself which spins the edges in opposite directions has infinitely many fixed points.

This will give us



fixed point of map ≡ periodic geodesic.

It is easy to show area preservation.

Two problems:

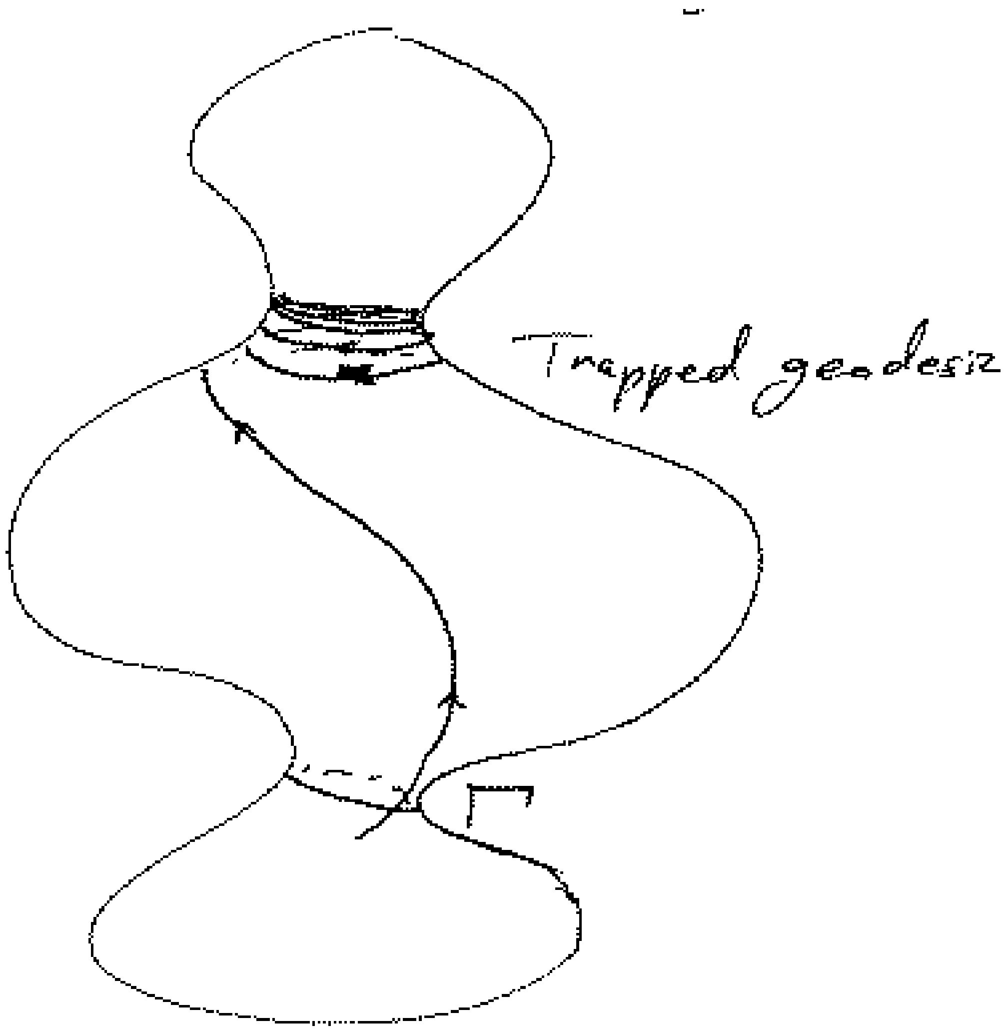
(1) do the edges of the cylinder spiral in different directions? No, not always.

(2) maybe a geodesic gets stuck and won't come back to Γ, so the map is not determined.

These problems are resolved

(1) by complex analysis

(2) by delicate geometry.



Theorem 2 [Bangert-Franks-Hingston]
The number of periodic geodesics on a potato with length less than L is $\geq C \cdot \frac{L}{\log L}$, where C is a positive constant.

$L/\log L$ comes from number theory.