Current state of the homological conjectures: Five talks of a VIGRE-funded minicourse at the University of Utah, in June, 2004

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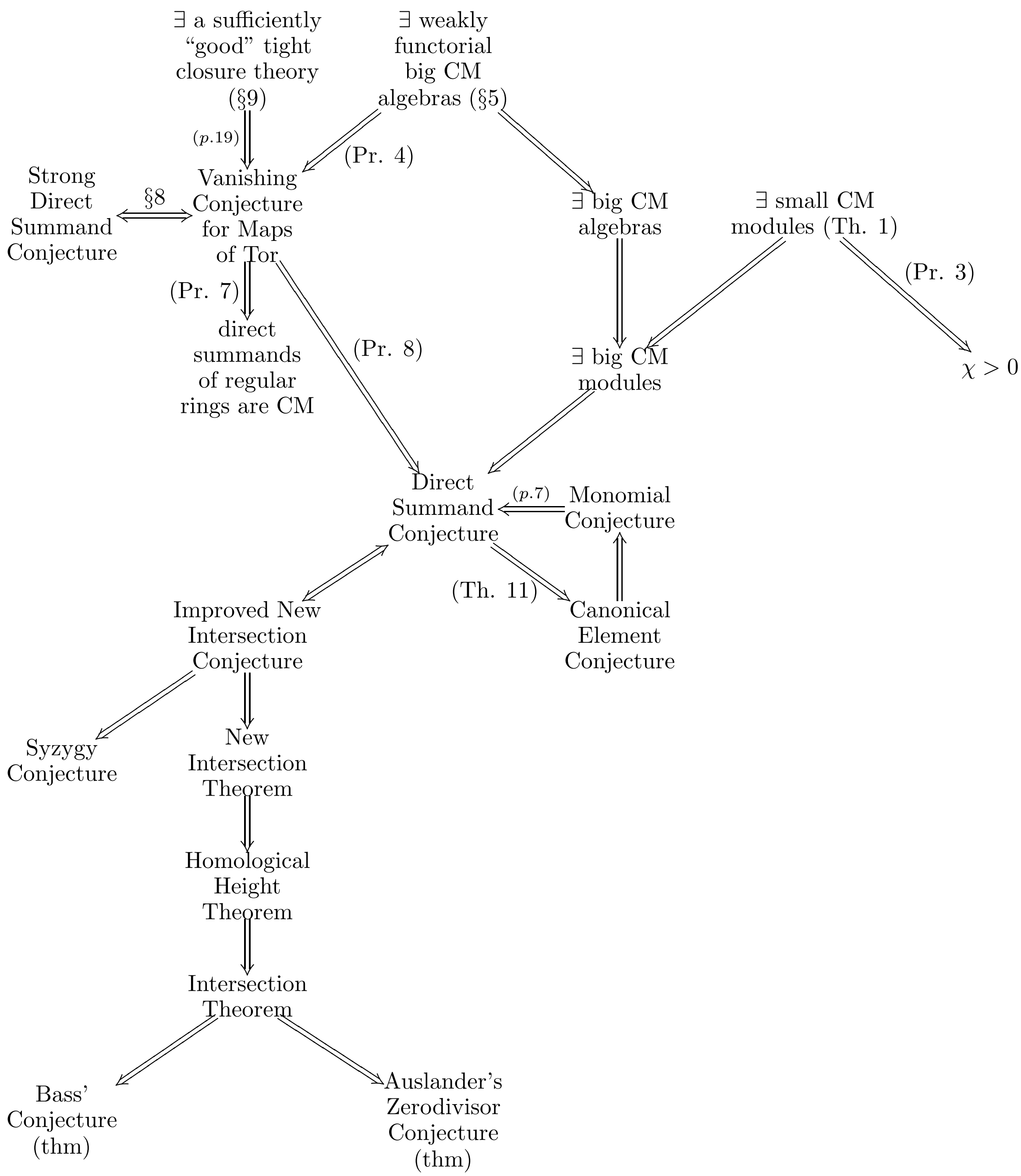
1 Homological Conjectures: a diagram



2 C-M modules, and the Vanishing Conjecture on Maps of Tor

Throughout these lectures, all C-M modules are assumed to be maximal C-M modules.

Definition. Let $(R, \mathfrak{m})$ be a local ring. An R-module, M, is a *big C-M module* if $\mathfrak{m}M \neq M$ and every s.o.p. is a regular M-sequence. M is a *small C-M module* if it is finitely generated, as well.

Note that in the above definition, it is enough for one s.o.p. to be a regular M-sequence, as this implies that every s.o.p is a regular M-sequence.

Conjecture 1 (Hochster). *($\exists$ small C-M modules) If R is an excellent, local ring, then R has a small C-M module.*

It is the opinion of the lecturer that this conjecture is generally false, though it has a chance of holding in char $p > 0$.

Example: If R is a complete local domain with dimension less than 3, then the normalization of R is a small C-M module.

Theorem 1. *[Hartshorne, Peskine-Szpiro, Hochster] Let R be a finitely generated $\mathbb{N}$-graded domain where $R_0 = K$, a perfect field of char $p > 0$, and $R_{\mathfrak{p}}$ is C-M for all prime $\mathfrak{p}$ except for where $\mathfrak{p} = \mathfrak{m}$, the homogeneous maximal ideal. Then R has a small, graded C-M module.*

Proof. Regard R as a module over itself via the eth iteration of the Frobenius map, $F^e : R \to {}^eR$ (which is a module-finite map). For M, a finitely generated R-module, the first non-vanishing $H^i_{\mathfrak{m}}(M)$ occurs for $i = \mathrm{depth}_{\mathfrak{m}}(M)$. Thus M is a small C-M module if and only if $H^i_{\mathfrak{m}}(M) = 0$, for all $i < d := \dim(R)$. Also, we have that $H^i_{\mathfrak{m}}({}^eM) = {}^eH^i_{\mathfrak{m}}(M)$. When M is such that $M_{\mathfrak{p}}$ is C-M for all $\mathfrak{p} \in \mathrm{Spec}(R) - \{\mathfrak{m}\}$, local duality gives that $H^i_{\mathfrak{m}}(M)$ has finite length.

Now let $n := \sum_{i<d} l(H^i_{\mathfrak{m}}(R))$. (Note that the above comment and our assumption on R gives that $n < \infty$.) For r, $0 \leq r < p^e$, let $M_r := \bigoplus_{i \equiv r (\mathrm{mod}\ p^e)} R_i$. Thus we have that ${}^eR \simeq \bigoplus_{0 \leq r < p^e} M_r$. We may choose e so that there are more $M_r \neq 0$ than n. Now $\bigoplus_{i<d} H^i_{\mathfrak{m}}({}^eR) = \bigoplus_{i<d} {}^eH^i_{\mathfrak{m}}(R)$ is a n-dimensional vector space. Since also

$$\bigoplus_{i<d} H^i_{\mathfrak{m}}({}^eR) = \bigoplus_{0 \leq r < p^e} (\bigoplus_{i<d} H^i_{\mathfrak{m}}(M_r)),$$

a dimension argument and the choice of e gives that $\bigoplus_{i<d} H^i_{\mathfrak{m}}(M_{r'}) = 0$ for some $M_{r'} \neq 0$. Thus $M_{r'}$ is a small C-M R-module. $\square$

From the above we get the following Corollary:

Corollary 2. *If R is a 3-dimensional, graded domain with char $p > 0$, then R has a small C-M module.*

We may consider the conjecture in another way. Let R be a complete local domain, where $A \subseteq R$ is a module-finite extension with A regular. Let M be an R-module. We have that

$$\begin{array}{rcc} M \text{ is a small C-M } R\text{-module} & \Leftrightarrow & M \text{ is a small C-M } A\text{-module} \\ & \Leftrightarrow & \operatorname{depth}_A(M) = \dim(A) = \operatorname{depth}_A(A) \\ & \Leftrightarrow & \operatorname{pd}_A(M) = 0 \\ & \Leftrightarrow & M \text{ is } A\text{-free.} \end{array}$$

Hence a small C-M R-module exists if and only if R embeds into $M_N(A)$, $N >> 0$, extending the natural embedding of A into $M_N(A)$ (as scalar diagonal matrices).

Proposition 3. *Conjecture 1 implies Serre's M_2 Conjecture ($\chi > 0$).*

Proof. Let R be regular, local and M, N be finitely generated R-modules with $\dim(M) + \dim(N) = \dim(R)$ and $l(M \otimes N) < \infty$. Given a filtration

$$0 \to M_1 \to M \to M_2 \to 0$$

for M, we have that $\chi(M, N) = \chi(M_1, N) + \chi(M_2, N)$. By filtering M and N by prime cyclic modules, we get that $\chi(M, N) = \sum_{i,j} \chi(R/\mathfrak{p}_i, R/\mathfrak{q}_j)$, where $\mathfrak{p}_i, \mathfrak{q}_j \in \operatorname{Spec}(R)$. Thus it's enough to prove that $\chi(R/\mathfrak{p}, R/\mathfrak{q}) > 0$, for $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$.

In assuming the validity of Conjecture 1, we have that $R/\mathfrak{p}$ has a small C-M module M' and $R/\mathfrak{q}$ has a small C-M module N'. M' will have a filtration with r copies, say, of $R/\mathfrak{p}$ (all other modules being of the form $R/\mathfrak{p}'$, $\mathfrak{p}' \supsetneq \mathfrak{p}$) and N' will have a filtration with s copies of $R/\mathfrak{q}$. Then $\chi(M', N') = rs \cdot \chi(R/\mathfrak{p}, R/\mathfrak{q})$, as all other terms vanish. From Serre, we have that $\chi(M', N') = l(M' \otimes N') > 0$, which completes the proof. $\square$

The following is more obviously a consequence of Conjecture 1.

Conjecture 2. *($\exists$ big C-M modules) If R is a local ring, then R has a big C-M module. (Via a reduction, we can assume that R is a complete domain.)*

Conjecture 2 has been proved in equal characteristic by Hochster in 1973. (See [Hoc75] for a proof.)

Conjecture 3. *($\exists$ weakly functorial big C-M algebras) Given a local map of complete local domains $R \to S$, there exists a commutative diagram:*

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

where B, C are quasi-local big C-M algebras and the maps are local.

Conjecture 3 holds in equal characteristic (Hochster and Huneke [HH92]) and in mixed characteristic if $\dim(R), \dim(S) \leq 3$. (Hochster [Hoc02], using Heitmann [Hei02].)

Conjecture 4. *(Vanishing Conjecture on Maps of* Tor*) For $R \to S \to T$, let R be regular, S be module-finite over R, and T be regular. If M is an R-module (not necessarily finitely generated), then $\operatorname{Tor}_i^R(M,S) \to \operatorname{Tor}_i^R(M,T)$ is 0 for $i \geq 1$.*

The validity of the Vanishing Conjecture on Maps of Tor would imply that:
i.) direct summands of regular rings are C-M, and
ii.) a regular ring is a direct summand of every module-finite extension.

Proposition 4. *($\exists$ weakly functorial big C-M algebras) $\Longrightarrow$ (Vanishing Conjecture on Maps of* Tor*).*

Before proceeding with the proof, we first need a lemma.

Lemma 5. *Let R be a regular local ring. An R-module, B, is a big C-M module if and only if B is faithfully flat over R.*

Proof. ($\Longleftarrow$) Suppose B is faithfully flat. Let $x_1, \ldots, x_k$ be part of a system of parameters. Thus $x_1, \ldots, x_k$ is a regular R-sequence, i.e. $K_{\cdot}(x_1, \ldots, x_k; R)$ is acyclic. Since B is flat, $K_{\cdot}(x_1, \ldots, x_k; B) = K_{\cdot}(x_1, \ldots, x_k; R) \otimes B$ is acyclic. Thus $x_1, \ldots, x_k$ is a regular B-sequence. By hypothesis, $B/\mathfrak{m}B = R/\mathfrak{m} \otimes_R B \neq 0$, thus $B \neq \mathfrak{m}B$.

($\Longrightarrow$) We want to show that for M, a finitely generated R-module, $\operatorname{Tor}_i^R(M,B) = 0$ for $i \geq 1$. If $i > \dim(R)$, $\operatorname{Tor}_i^R(M,B) = 0$, since $\operatorname{pd}_R(M) \leq \dim(R)$. Assume that for some $i \geq 1$, $\operatorname{Tor}_{i+1}^R(M,B) = 0$, for all finitely generated R-modules, M. M has a prime cyclic filtration, so it is sufficient to consider the case $M = R/\mathfrak{p}$. Choose a maximal regular R-sequence in $\mathfrak{p}$, $x_1, \ldots, x_k$. We have a short exact sequence:

$$0 \to R/\mathfrak{p} \to R/(x_1, \ldots, x_k) \to N \to 0,$$

where N is the cokernel. Thus we have:

$$\operatorname{Tor}_{i+1}^R(N,B) \to \operatorname{Tor}_i^R(R/\mathfrak{p},B) \to \operatorname{Tor}_i^R(R/(x_1, \ldots, x_k),B).$$

By reverse induction, $\operatorname{Tor}_{i+1}^R(N,B) = 0$; and $\operatorname{Tor}_i^R(R/(x_1, \ldots, x_k),B) = 0$, since $x_1, \ldots, x_k$ is also a regular B-sequence. Hence $\operatorname{Tor}_i^R(R/\mathfrak{p},B) = 0$. Finally, since $\mathfrak{m}B \neq B$, B is a faithfully flat R-module. $\square$

We continue with the proof of Proposition 4.

Proof. (of Proposition 4) For $R \to S \to T$, let R be regular, S be module-finite over R, T be regular, and let M be an R-module. Using easy reductions, we may assume that M is finitely generated, R, T are complete and local, and S is a local domain. By assumption, we get a commutative diagram:

$$\begin{tikzcd}
 & B \arrow[r] & C \\
R \arrow[ur] \arrow[r] & S \arrow[u] \arrow[r] & T, \arrow[u]
\end{tikzcd}$$

where B and C are quasi-local big C-M algebras over S and T, respectively. Note that we can regard B as a big C-M module over R, since an s.o.p. in R is an s.o.p. in S. From the above lemma, B is faithfully flat over R and C is faithfully flat over T. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Tor}_i^R(M,S) \arrow[r] \arrow[dr, "\phi"'] & \mathrm{Tor}_i^R(M,B) \arrow[r] & \mathrm{Tor}_i^R(M,T) \\
 & \mathrm{Tor}_i^R(M,T). \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Since B is faithfully flat over R, $\mathrm{Tor}_i^R(M,B) = 0$. Also, since C is faithfully flat over T, the map ψ is injective. Hence ϕ is the zero map, which completes the proof. $\square$

3 Direct summands of regular rings, the Direct Summand Conjecture, and the Monomial Conjecture

Consider a field K having char 0. Let $G := GL(t, K)$ and consider $T := K[x_{ij}, y_{jk}]$, where $X = [x_{ij}]$, $Y = [y_{jk}]$ are $r \times t$, $t \times s$ matrices, respectively. For $\alpha \in G$, the assignments $X \mapsto X\alpha^{-1}$, $Y \mapsto \alpha Y$ give an action of G on T.

Let $T^G = K[Z]$, where $Z = XY$, but the entries of Z are not algebraically independent. So $T^G = K[U]/I_{t+1}(U)$, where $U = [u_{ik}]$ is an $r \times s$ matrix.

Theorem 6. *T^G is a direct summand of T as a T^G-module.*

This example serves to motivate the next conjecture.

Conjecture 5. *(Direct summands of regular rings are C-M) For $R \to T$, if R is a direct summand of T as an R-module and T is regular, then R is C-M.*

This conjecture is open for mixed characteristic. As was mentioned in the previous lecture, the conjecture is implied by the Vanishing Conjecture on Maps of Tor.

Proposition 7. *(Vanishing Conjecture on Maps of* Tor*) $\Longrightarrow$ (Direct summands of regular rings are C-M).*

Proof. Via an easy reduction, we may assume that R is local and complete. Choose $A \subseteq R$, where R is module-finite over A, A regular. Choose an s.o.p. $x_1, \ldots, x_d$ in A; let $M := A/(x_1, \ldots, x_d)$. The Vanishing Conjecture gives that the maps $\mathrm{Tor}_i^A(M,R) \to \mathrm{Tor}_i^A(M,T)$ are 0 for $i \geq 1$. But $T = R \oplus_A \heartsuit$. Thus $\mathrm{Tor}_i^A(M,R) = 0$, for $i \geq 1$. So $x_1, \ldots, x_d$ is a regular R-sequence, hence R is C-M. $\square$

Conjecture 6. *(Direct Summand Conjecture) For $A \subseteq R$, if R is module-finite over A and A is regular, then A is a direct summand of R as an A-module.*

Notes. i.) Kill a minimal prime, $\mathfrak{p}$, in R disjoint from $A - \{0\}$. Consider the diagram:

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[dr, hook, "g"'] & R \arrow[d] \\ & R/\mathfrak{p} \end{tikzcd}$$

If g splits, so does f, hence we may assume that R is a domain.

ii.) If $\mathbb{Q} \subseteq A$, then when you consider a diagram:

$$\begin{array}{ccc} K & \subseteq & L \\ \uparrow & & \\ A & \subseteq & S, \end{array}$$

we have that L/K is a finite field extension, say $[L : K] = h$. So the map $\frac{1}{h}Tr_{L/K} : R \to A$ is a splitting (values are in A when A is normal).

iii.) Notice that:

$$\begin{aligned} A \overset{j}{\hookrightarrow} R \text{ splits} \quad &\Leftrightarrow \quad \mathrm{Hom}_A(R, A) \to \mathrm{Hom}_A(A, A) \text{ is onto} \\ &\Leftrightarrow \quad \exists f : R \to A \text{ such that} f \circ j = id_A. \end{aligned}$$

iv.) Since for finitely generated modules, localization commutes with Hom, and Hom commutes with $\_ \otimes M$ for M flat, we may assume that our rings are complete regular local.

v.) If $A \hookrightarrow R$ splits ($R = A \oplus \heartsuit$), then every ideal of A is contracted from R. For $I \subseteq A$, $IR = IA \oplus I\heartsuit$, so $I = (IA \oplus I\heartsuit) \cap (A \oplus 0)$.

vi.) Given a sequence of $\mathfrak{m}$-primary irreducible ideals I_n, cofinal with powers of $\mathfrak{m}$, $A \hookrightarrow R$ splits if and only if $I_n R \cap A = I_n$ for all n. So if $x_1, \ldots, x_d$ is an s.o.p. for A, it is sufficient to show that $I_n := (x_1^n, \ldots, x_d^n)$ is contractible for all n. So in A, if $x_1, \ldots, x_d$ are generators of $\mathfrak{m}_A$, the socle element in A/I_n is represented by $x_1^{n-1} \cdots x_d^{n-1}$. Hence I_n is contracted from R if and only if $x_1^{n-1} \cdots x_d^{n-1} \notin I_n R$. This leads to the following conjecture.

Conjecture 7. *(Monomial Conjecture) Let R be local. If $x_1, \ldots, x_d$ is an s.o.p., then $x_1^{n-1} \cdots x_d^{n-1} \notin (x_1^n, \ldots, x_d^n)R$ for all n.*

This conjecture is open in mixed characteristic where dim ≥ 4. The note above gives that:

$$\text{(Monomial Conjecture)} \Longrightarrow \text{(Direct Summand Conjecture)}$$

Proposition 8. *(Vanishing Conjecture on Maps of* Tor*)* $\Longrightarrow$ *(Direct Summand Conjecture).*

Proof. (Proceed by proving contrapositive.) Let $A \hookrightarrow R$ be a module-finite extension, where A is a complete regular local ring, and R is a complete local domain. Let $x_1, \ldots, x_d$ be a minimal generating set of $\mathfrak{m}_A$. Suppose there exist $r_i \in R$ such that $x_1^{n-1} \cdots x_d^{n-1} - \sum_{i=1}^d r_i x_i^n = 0$. Setting $T := R/\mathfrak{m}_R$, we have maps $A \hookrightarrow R \to T$. Let $M := A/(x_1^n, \ldots, x_d^n, x_1^{n-1} \cdots x_d^{n-1})$. Now

$$\operatorname{Tor}_1^A(M, R) = \frac{\{\text{relations on } x_1^n, \ldots, x_d^n, x_1^{n-1} \cdots x_d^{n-1} \text{ in } R\}}{\{\text{relations on } x_1^n, \ldots, x_d^n, x_1^{n-1} \cdots x_d^{n-1} \text{ in } A\}}.$$

From the above, $(1, -r_1, \ldots, -r_d)$ represents a relation in R. So under the map $\operatorname{Tor}_1^A(M, R) \to \operatorname{Tor}_1^A(M, T)$, $\overline{(1, -r_1, \ldots, -r_d)} \mapsto \overline{(1, 0, \ldots, 0)} \neq 0$, hence the map is non-zero. $\square$

Conjecture 8. *(Strong Direct Summand Conjecture) For $A \subseteq R$, suppose R is a module-finite domain over A, where A is regular local. Let x be a regular parameter ($x \in \mathfrak{m}_A - \mathfrak{m}_A^2$) and $\mathfrak{q}$ be a height 1 prime in R lying over xA. Then xA is a direct summand of $\mathfrak{q}$ as an A-module.*

As one should expect, this implies the Direct Summand Conjecture. $xA \subset xR \subset \mathfrak{q}$ gives that xA is a direct summand of xR, thus A is a direct summand of R.

Conjecture 9. *(Syzygy Conjecture) Suppose that R is regular and M is a finitely generated R-module (or that R is C-M and* $\operatorname{pd}(M) < \infty$*). If any k^{th} syzygy of M is not free, then it has rank $\geq k$.*

This is known in equal characteristic (Evans-Griffith [EG82]), and uses the existence of big C-M modules. In fact, it is enough to know the Improved New Intersection Theorem.

4 The Canonical Element Conjecture

[1]

Let $(R, \mathfrak{m}, k)$ be local, $\dim(R) = d$, and consider a free resolution of k:

$$\cdots \to R^{b_d} \to R^{b_{d-1}} \to \cdots \to R^{b_0} \to k \to 0.$$

So this yields an exact sequence:

$$\varepsilon : 0 \to \operatorname{syz}^d(k) \to R^{b_{d-1}} \to \cdots \to R^{b_0} \to k \to 0.$$

Using Yoneda's definition of Ext, $\operatorname{Ext}_R^i(A, B)$ can be identified with the set of equivalence classes of exact sequences,

$$0 \to B \to M_{i-1} \to \cdots \to M_0 \to A \to 0,$$

[1] All material in this Section may be found in the lecturer's paper [Hoc83].

joining B to A.

We have an element $\overline{\varepsilon} \in \mathrm{Ext}_R^d(k, \mathrm{syz}^d(k))$ representing the equivalence class of the exact sequence above. Since

$$\lim_{\overrightarrow{t}} \mathrm{Ext}_R^d(R/\mathfrak{m}^t, \mathrm{syz}^d(k)) = H_{\mathfrak{m}}^d(\mathrm{syz}^d(k)),$$

$\overline{\varepsilon} \mapsto \eta \in H_{\mathfrak{m}}^d(\mathrm{syz}^d(k))$. η is called the canonical element in $H_{\mathfrak{m}}^d(\mathrm{syz}^d(k))$.

Conjecture 10. *(Canonical Element Conjecture)* $\eta \neq 0$.

Notes. i.) **Theorem.** If R has a big C-M module, then $\eta \neq 0$.

ii.) For a ring R such that R maps to a local ring R_1 where s.o.p.'s map to s.o.p.'s, if $\eta_{R_1} \neq 0$ then $\eta_R \neq 0$.

iii.) We may assume that R is a complete normal local domain.

The Canonical Element Conjecture affords an alternative formulation. Consider a resolution of k, $F_.$, and an s.o.p. $x_1, \ldots, x_d$. We have a diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & R^{b_{d+1}} \arrow[r] & R^{b_d} \arrow[r] & R^{b_{d-1}} \arrow[r] & \cdots \arrow[r] & R^{b_0} \arrow[r] & k \arrow[r] & 0 \\
 & 0 \arrow[r] & R \arrow[r] \arrow[u, dashed, "\phi"] & R^d \arrow[r] \arrow[u, dashed] & \cdots \arrow[r] & R \arrow[r] \arrow[u, dashed] & R/(x_1, \ldots, x_d) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Since the top row is exact and free modules are projective, there exist maps which makes the diagram commute. Notice that ϕ induces a map $R \to \mathrm{syz}^d(k)$. The element in $\mathrm{syz}^d(k)$ that is the image of 1 is only determined modulo $(x_1, \ldots, x_d)\mathrm{syz}^d(k)$, since we may shift up to homotopy. Hence we get a well-defined element $\eta_1 \in \mathrm{syz}^d(k)/(x_1, \ldots, x_d)\mathrm{syz}^d(k)$. Since $x_1^t, \ldots, x_d^t$ is also an s.o.p., we can repeat the process to get an element $\eta_t \in \mathrm{syz}^d(k)/(x_1^t, \ldots, x_d^t)\mathrm{syz}^d(k)$.

We get a map

$$\frac{\mathrm{syz}^d(k)}{(x_1, \ldots, x_d)\mathrm{syz}^d(k)} \overset{x_1^{t-1} \cdots x_d^{t-1}}{\longrightarrow} \frac{\mathrm{syz}^d(k)}{(x_1^t, \ldots, x_d^t)\mathrm{syz}^d(k)},$$

where $\eta_1 \mapsto \eta_t$. Taking direct limits, we have

$$\lim_{\overrightarrow{t}} \frac{\mathrm{syz}^d(k)}{(x_1^t, \ldots, x_d^t)\mathrm{syz}^d(k)} = H_{\mathfrak{m}}^d(\mathrm{syz}^d(k)),$$

giving an element $\eta \in H_{\mathfrak{m}}^d(\mathrm{syz}^d(k))$. This η is the same element found previously.

Note. An equivalent conjecture is that the map ϕ is never zero for any choice of complex maps or system of parameters.

Theorem 9. *The Direct Summand Conjecture implies the Monomial Conjecture.*

First, we may assume that we are in the domain case, so each x_i is a nonzerodivisor. Hence, although the Koszul complex is *a priori* defined by the second line in the following commutative diagram, we may as well define it by the first line.

$$
\begin{tikzcd}
K_{\cdot}(\mathbf{x};R) \cong \bigotimes_i \big(0 \arrow[r] & x_i R \arrow[r] & R \arrow[r] & 0 \big) \\
0 \arrow[r] & R \arrow[r, "x_i"] \arrow[u, "x_i", "\cong"'] & R \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}
$$

Moreover, with this identification, $x_iR \otimes x_jR$ maps to x_ix_jR, so the Koszul complex is of the form

$$K_{\cdot}(\mathbf{x};R) = 0 \to x_1 \cdots x_d R \to \bigoplus_j x_1 \cdots \widehat{x_j} \cdots x_d R \to \cdots \to \bigoplus_j R x_j \to R \to 0. \tag{1}$$

Now, let R^+ be the biggest integral domain that is integral over R. This is unique, up to non-unique isomorphism, and may alternately be defined as the integral closure of R in an algebraic closure of the fraction field. For any positive integer p, the sequence $\mathbf{x}^{1/p} = x_1^{1/p}, \ldots, x_d^{1/p}$ exists in $R[x_1^{1/p}, \ldots, x_d^{1/p}] \subseteq R^+$, so since R^+ is an integral domain, we get from (1) the following inclusion of $K_{\cdot}(\mathbf{x};R^+)$ as a subcomplex of $K_{\cdot}(\mathbf{x}^{1/p};R^+)$:

$$
\begin{tikzcd}
0 \arrow[r] & x_1 \cdots x_d R^+ \arrow[r] \arrow[d, hook] & \bigoplus_j x_1 \cdots \widehat{x_j} \cdots x_d R^+ \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & \bigoplus_j R^+ x_j \arrow[r] \arrow[d, hook] & R^+ \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (x_1 \cdots x_d)^{1/p} R^+ \arrow[r] & \bigoplus_j x_1^{1/p} \cdots \widehat{x_j^{1/p}} \cdots x_d^{1/p} R^+ \arrow[r] & \cdots \arrow[r] & \bigoplus_j R^+ x_j^{1/p} \arrow[r] & R^+ \arrow[r] & 0
\end{tikzcd}
\tag{2}$$

Now *assume further* that the residue field of R has positive prime characteristic p. With the above diagram in mind, for $x \in R^+$ write

$$(x^\infty) := \bigcup_n x^{1/p^n} R^+.$$

Taking the union (really a direct limit) of containments of Koszul complexes as in (2), we get the following flat, though no longer free, complex over R^+:

$$\mathcal{K}^d := \left(0 \to ((x_1 \cdots x_d)^\infty) \to \bigoplus_j ((x_1 \cdots \widehat{x_j} \cdots x_d)^\infty) \to \cdots \to \bigoplus_j R^+(x_j^\infty) \to R^+ \to 0 \right). \tag{3}$$

Note that none of this depends on $x_1, \ldots, x_d$ being a *full* system of parameters. Now we come to the following important

Lemma 10. *If $x_1, \ldots, x_d$ is part of a system of parameters for R, then $\mathcal{K}^d$ is acyclic in characteristic p, and also in mixed characteristic if $x_1 = p$.*

Proof in mixed characteristic. We have $\mathcal{K}^d = \bigotimes_i (0 \to (x_i^\infty) \to R^+ \to 0)$. We proceed by induction on d, noting that the case $d = 1$ is obvious.

By induction, then, $\mathcal{K}^{d-1} = \bigotimes_{i=1}^{d-1} (0 \to (x_i^\infty) \to R^+ \to 0)$ is acyclic, with augmentation $R^+/(x_1^\infty, \ldots, x_{d-1}^\infty)$.

Tensoring with the complex $0 \to (x_d^\infty) \to R^+ \to 0$, we get the double complex

$$\begin{tikzcd} 0 \arrow[d] \\ (x_d^\infty) \otimes \mathcal{K}^{d-1} \arrow[d] \\ \mathcal{K}^{d-1} \arrow[d] \\ 0 \end{tikzcd}$$

whose total complex is $\mathcal{K}^d$. Thus, the mapping cone construction provides a short exact sequence of complexes

$$0 \to \mathcal{K}^{d-1} \to \mathcal{K}^d \to \left((x_d^\infty) \otimes \mathcal{K}^{d-1}\right)[-1] \to 0$$

giving a long exact sequence in homology, a typical row of which looks like:

$$\cdots \to H_i(\mathcal{K}^{d-1}) \to H_i(\mathcal{K}^d) \to H_{i-1}((x_d^\infty) \otimes \mathcal{K}^{d-1}) \to \cdots$$

Since $\mathcal{K}^{d-1}$ is acyclic and (x_d^∞) is a flat R^+-module, we always have $H_i(\mathcal{K}^{d-1}) = 0$ for all $i \geq 1$ and

$$H_{i-1}((x_d^\infty) \otimes \mathcal{K}^{d-1}) \cong (x_d^\infty) \otimes H_{i-1}(\mathcal{K}^{d-1}) = 0.$$

for all $i \geq 2$. Thus, when $i \geq 2$, the long exact sequence above shows that $H_i(\mathcal{K}^d) = 0$, and when $i = 1$ we get the exact sequence

$$0 \to H_1(\mathcal{K}^d) \to (x_d^\infty) \otimes H_0(\mathcal{K}^{d-1}) \to H_0(\mathcal{K}^{d-1}),$$

so to finish the proof of the lemma, it's enough to show that the last map is injective. That is, we want to show that

$$(x_d^\infty) \otimes \frac{R^+}{(x_1^\infty, \ldots, x_{d-1}^\infty)} \to \frac{R^+}{(x_1^\infty, \ldots, x_{d-1}^\infty)}$$

is injective. Since (x_d^∞) is flat, what we need to show is:

$$(x_d^\infty) \cap (x_1^\infty, \ldots, x_{d-1}^\infty) = (x_d^\infty)(x_1^\infty, \ldots, x_{d-1}^\infty). \tag{4}$$

So, let $u \in (x_d^\infty) \cap (x_1^\infty, \ldots, x_{d-1}^\infty)$. Recall that $x_1 = p$. Then

$$u = s_1 p^{1/q_1} + s_2 x_2^{1/q_2} + s_3 x_3^{1/q_3} + \cdots + s_{d-1} x_{d-1}^{1/q_{d-1}},$$

where $q_j = p^{n_j}$ are powers of p. Then

$$\left(u^{1/p} - s_1^{1/p}p^{1/pq_1} - s_2^{1/p}x_2^{1/pq_2} - \cdots - s_{d-1}x_{d-1}^{1/pq_{d-1}}\right)^p$$
$$= (u - s_1p^{1/q_1} - s_2x_2^{1/q_2} - \cdots - s_{d-1}x_{d-1}^{1/q_{d-1}}) + pv = pv$$

for some $v \in R^+$. The reason is that when expanding out the p'th power of a sum of monomials, whenever p is a prime, the resulting monomials in the expanded sum are of two forms: the p'th powers of the original monomials, and multiples of p. We have collected all the multiples of p into one term: pv.

Taking p'th roots in the displayed equation,

$$u^{1/p} - \sum_{j=1}^{d-1} s_j^{1/p}x_j^{1/pq_j} = p^{1/p}v^{1/p},$$

so that

$$\begin{aligned} u^{1/p} &= p^{1/p}v^{1/p} + s_1^{1/p}p^{1/pq_1} + \sum_{j=2}^{d-1} s_j^{1/p}x_j^{1/pq_j} \\ &= \left((p^{1/pq_1})^{q_1-1}v^{1/p} + s_1^{1/p}\right)p^{1/pq_1} + \sum_{j=2}^{d-1} s_j^{1/p}x_j^{1/pq_j} \\ &\in (x_1^\infty, \ldots, x_{d-1}^\infty). \end{aligned}$$

Since $u^{1/p} \in (x_d^\infty)$ and $u^{1/p} \in (x_1^\infty, \ldots, x_{d-1}^\infty)$, it follows that

$$u = u^{1/p}(u^{1/p})^{p-1} \in (x_d^\infty)(x_1^\infty, \ldots, x_{d-1}^\infty),$$

as was to be shown. $\square$

Theorem 11. *The direct summand conjecture implies the canonical element conjecture in residual characteristic $p > 0$.*

Proof for complete mixed characteristic local domains $(R, \mathfrak{m}, K)$. [2] Extend p to a system of parameters $x_1 = p, x_2, \ldots, x_d$ for R. Given a counterexample to the CEC, there is some t such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "0"] & R^d \arrow[r] \arrow[d] & \cdots \arrow[r] & R^{\binom{d}{2}} \arrow[r] \arrow[d] & R^d \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/(x_1^t, \ldots, x_d^t) \arrow[r] \arrow[d] & 0 \\ \cdots \arrow[r] & R^{b_d} \arrow[r] & R^{b_{d-1}} \arrow[r] & \cdots \arrow[r] & R^{b_2} \arrow[r] & R^{b_1} \arrow[r] & R \arrow[r] & K \arrow[r] & 0, \end{tikzcd}$$

where the top row is the Koszul complex on $x_1^t, \ldots, x_d^t$, the bottom row is the minimal free resolution of K over R, the rightmost vertical map is the

[2]Formally the same proof works in characteristic p.

canonical surjection, and the other vertical maps are the liftings guaranteed by homological algebra since (before the augmentation) the bottom complex is acyclic and the top one is free.

On the other hand, letting $K^+ = R^+/(x_1^\infty, \ldots, x_d^\infty)$, we have an injection $K \hookrightarrow K^+$, which, since $\mathcal{K}^d$ from the Lemma is acyclic, induces a map of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & R^{b_d} \arrow[r] \arrow[d] & R^{b_{d-1}} \arrow[r] \arrow[d] & \cdots \arrow[r] & R^{b_1} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (\mathcal{K}^d)_d \arrow[r] & (\mathcal{K}^d)_{d-1} \arrow[r] & \cdots \arrow[r] & (\mathcal{K}^d)_1 \arrow[r] & R^+ \arrow[r] & K^+ \arrow[r] & 0
\end{tikzcd}$$

from the resolution of K to this "funny" Koszul complex.

Composing the above two maps of complexes, we get a map of complexes

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "a_d"] \arrow[d, "0"'] & R^d \arrow[r] \arrow[d] & \cdots \arrow[r] & R^d \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/(x_1^t, \ldots, x_d^t) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & ((x_1 \cdots x_d)^\infty) \arrow[r] & (\mathcal{K}^d)_{d-1} \arrow[r] & \cdots \arrow[r] & (\mathcal{K}^d)_1 \arrow[r] & R^+ \arrow[r] & K^+ \arrow[r] & 0
\end{tikzcd}$$

where $a_d = \begin{bmatrix} x_1^t \\ \vdots \\ x_d^t \end{bmatrix}$ and the last vertical map is the canonical inclusion. However, there is also the standard way to map the Koszul complex on $x_1^t, \ldots, x_d^t$ to the "funny" one, as a subcomplex map:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "a_d"] \arrow[d, hook, "x_1^t \cdots x_d^t"'] & R^d \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & R^d \arrow[r] \arrow[d, hook] & R \arrow[r] \arrow[d, hook] & R/(x_1^t, \ldots, x_d^t) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & ((x_1 \cdots x_d)^\infty) \arrow[r] & (\mathcal{K}^d)_{d-1} \arrow[r] & \cdots \arrow[r] & (\mathcal{K}^d)_1 \arrow[r] & R^+ \arrow[r] & K^+ \arrow[r] & 0
\end{tikzcd}$$

Since the map from K to K^+ is the same in both of the above commutative diagrams, the maps of complexes must be homotopic. In particular, the difference of the first vertical maps must factor through a_d, which implies that

$$x_1^t \cdots x_d^t \in (x_1^t, \ldots, x_d^t)(x_1 \cdots x_d)^{1/q}$$

in R^+ for some large power q of p.

Note that this all happens in a module-finite extension S of R. Letting $y_j = x_j^{1/q}$, we have $x_j = y_j^q$, and so

$$y_1^{qt} \cdots y_d^{qt} \in (y_1^{qt+1}, \ldots, y_d^{qt+1}).$$

Moreover, $y_1, \ldots, y_d$ live in a regular subring $V[\![y_2, \ldots, y_d]\!]$ of S, where $\mathfrak{m}_V = (p^{1/q})V$. This contradicts the direct summand conjecture for the regular subring $V[\![y_2, \ldots, y_d]\!]$ of S. $\square$

Remarks:

- Thus, we have:

$$\begin{pmatrix}\text{Direct summand conjecture for} \\ \text{regular local rings of the form} \\ V[\![y_2,\ldots,y_d]\!], \mathfrak{m}_V = (p^{1/q})\end{pmatrix} \Rightarrow \text{CEC} \Rightarrow \text{Monomial conj.} \Rightarrow \begin{matrix}\text{Direct summand} \\ \text{conjecture.}\end{matrix}$$

- One can use this to show that the direct summand conjecture is equivalent to the assertion that $H^d_{\mathfrak{m}}(A^+) \neq 0$ for rings of the form $A = V[\![y_2,\ldots,y_d]\!]$, $\mathfrak{m}_V = (p^{1/q})$.

- In fact the DSC follows if it is true for $V[\![y_2,\ldots,y_d]\!]$, where $\mathfrak{m}_V = (p)$.

- Note that for R a complete local domain of positive residual characteristic, $\mathfrak{m}_{R^+} = (x_1^\infty,\ldots,x_d^\infty)$. Hence, $R^+/\mathfrak{m}_{R^+}$ has a *finite flat resolution* over R^+.

- The argument in Theorem 11 may be used to show that the existence of a big Cohen-Macaulay module implies the direct summand conjecture. To see this, let B be a big Cohen-Macaulay R-module. In the proof of Theorem 11, replace t with 1 and replace the "funny Koszul complex" from Lemma 10 with the actual Koszul complex $K_.(\mathbf{x}; B)$. Then let $u \in B \setminus \mathfrak{m}_R B$, and arrange it so that $(x_1,\ldots,x_d)u = 0$. The same homotopy argument as before puts $u \in (x_1,\ldots,x_d)B \subseteq \mathfrak{m}_R B$, giving the required contradiction.

5 Weak functoriality of big Cohen-Macaulay algebras

If $R \to S$ is a map of Noetherian local domains, then there is a map from R^+ to S^+ making the following diagram commute:

$$\begin{tikzcd} R^+ \arrow[r, dashed] & S^+ \\ R \arrow[u, hook] \arrow[r] & S \arrow[u, hook] \end{tikzcd}$$

That is, $(\ )^+$ is "weakly functorial" on Noetherian local domains. Here's the reason:

First note that any map $R \to S$ factors into an injection and a surjection, so it suffices to fill in diagrams of the form

$$\begin{tikzcd} R^+ \arrow[r, dashed] & R_1^+ \arrow[r, dashed] & S^+ \\ R \arrow[u, hook] \arrow[r, two heads] & R_1 \arrow[u, hook] \arrow[r, hook] & S \arrow[u, hook] \end{tikzcd}$$

Filling in $R_1^+ \to S^+$ works, since any algebraic closure of the fraction field of S will contain an algebraic closure of the fraction field of R_1, and the restriction of this inclusion to R_1^+ certainly lands in S^+.

As for $R \twoheadrightarrow R_1 = R/Q$, $Q \in \operatorname{Spec} R$, since R^+ is integral over R, there is a prime Q^+ of R^+ lying over Q, so that R/Q injects into R^+/Q^+. Moreover, every prime splits over R^+, so $R^+/Q^+ \cong (R/Q)^+ \cong R_1^+$.

Since R^+ is a big Cohen-Macaulay algebra in characteristic p, we immediately get weak functoriality of big Cohen-Macaulay algebras in characteristic p. Then by "reduction to characteristic p" (11 pages!), weak functoriality of big Cohen-Macaulay algebras in equal characteristic zero follows as well.

6 Uniform annihilation of (co)homology: "colon-killers"

[HH93] Suppose R is pure of dimension d, local, xcellent, and R_c is Cohen-Macaulay for some element $c \in R$. Then one can show that c has a power c^N that kills all higher Koszul homology $H_i(x_1, \ldots, x_k; R)$, $i \geq 1$, whenever $x_1, \ldots, x_k$ is part of any system of parameters for R. In consequence, c^N kills the quotient

$$\frac{(x_1, \ldots, x_k) : x_{k+1}}{(x_1, \ldots, x_k)}$$

whenever $x_1, \ldots, x_{k+1}$ is part of a system of parameters.

Since the N is independent of the system of parameters chosen, it works just as well for $x_1^t, \ldots, x_{k+1}^t$ for any t, so in fact c^N kills $H^i_{(x_1,\ldots,x_k)}$ for all $i < d = dim R$. The ideal generated by these "good" c's has height ≥ 2.

Now, let R be a complete local domain. By Cohen structure theorems, it is module-finite and torsion free over a regular local domain A. Letting h be the torsion-free rank of R over A, we can pick an inclusion $A^{\oplus h} \subseteq R$. Then $R/A^{\oplus h}$ is an A-torsion module, so we can pick $c \in A \setminus \{0\}$ such that $cR \subseteq A^{\oplus h}$. *Then c is a colon-killer for all systems of parameters for R in A.*

Proof. Let $x_1, \ldots, x_d$ be a system of parameters for R in A and $k < d$. Suppose $x_{k+1}u \in (x_1, \ldots, x_k)R$. We want to show that $cu \in (x_1, \ldots, x_k)R$. We have $cu \in A^{\oplus h}$ and

$$x_{k+1}(cu) \in (x_1, \ldots, x_k)(cR) \subseteq (x_1, \ldots, x_k)A^{\oplus h}.$$

But the x'es are a regular sequence on A, so

$$cu \in (x_1, \ldots, x_k)A^{\oplus h} \subseteq (x_1, \ldots, x_k)R.$$

□

Suppose A is a complete regular local ring. Then the direct summand conjecture for A is equivalent to the existence of a nonzero A-linear map $A^+ \to A$. This is because $\operatorname{Hom}_A(A^+, A)$ is Matlis-dual to $H^d_{\mathfrak{m}}(A^+)$.

7 Superheight

(Reference: [Hoc81])

For a proper ideal $I \subset R$, we define the *superheight* of I to be

$$\operatorname{superheight} I := \sup\{\operatorname{ht} IS \mid S \text{ Noetherian }, IS \neq S\}.$$

For example, let $R = k[X, Y, U, V]/(XY - UV) = k[x, y, u, v]$, letting lower-case letters denote the image of the upper-case variables in R, and set $\mathfrak{p} = (x, u)$. Then the *height* of $\mathfrak{p}$ is 1, since $\dim R = 3$ and $R/\mathfrak{p} \cong k[Y, V]$, so $\dim R/\mathfrak{p} = 2$. However, letting $S = R/(y, v) \cong k[X, U]$, we have $\operatorname{ht}(\mathfrak{p}S) = 2$, so $\operatorname{superheight} \mathfrak{p} \geq 2$. On the other hand, *Krull's Principal ideal theorem* can be restated as follows: $\operatorname{superheight}(X_1, \ldots, X_n) = n$ in $\mathbb{Z}[X_1, \ldots, X_n]$. Thus, in our case, this shows that $\operatorname{superheight} \mathfrak{p} \leq 2$, so that $\operatorname{superheight} \mathfrak{p} = 2$.

In terms of superheight, Serre's theorem implies that in a regular ring R, $\operatorname{superheight} P = \operatorname{ht} P$ for all $P \in \operatorname{Spec} R$.

We can also state the Monomial Conjecture in terms of superheight. Namely, let

$$R = \frac{\mathbb{Z}[X_1, \ldots, X_d, Y_1, \ldots, Y_d]}{(X_1^t X_2^t \cdots X_d^t - \sum_j Y_j X_j^{t+1})}$$

and let $P = (x_1, \ldots, x_d) \subseteq R$. Then the Monomial Conjecture states that the superheight of P is always $d - 1$.

To see this, note first that the height of P in R is $d-1$, and P has d generators, so the superheight must be either $d-1$ or d. Now, suppose that the superheight is d. Then localize at a minimal prime over P which has height d, and the x'es become a system of parameters which violate the monomial conjecture. On the other hand, any counterexample to the monomial conjecture occurs as an image of such an R and P, and the height of the image of P in the counterexample ring is d, so the superheight of P in R must be d.

Similarly, many of the homological conjectures can be stated in terms of superheight.

8 The vanishing conjecture for maps of Tor is equivalent to Nandini Ranganathan's "Strong Direct Summand Conjecture"

Setup: Let

$$A \subseteq R \to T$$

be maps of Noetherian rings such that R is module-finite and torsion-free over A, where A is a regular local ring, and let M be a finitely generated A-module. The Vanishing Conjecture for Maps of Tor would say that the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, T)$ is zero.

First reduction: Replace M by $\operatorname{syz}^1 M$ repeatedly, so that *we may assume that* $i = 1$.

Second reduction: There is a presentation of M as the quotient of a free A-module:

$$M = \frac{Au_1 + \cdots + Au_s}{\mathrm{Span}\,\{\sum_{j=1}^{s} a_{ij}u_j \mid i = 1, \ldots, h\}}$$

By replacing A by $A[\mathbf{u}] = A[u_1, \ldots, u_s]$, R by $R[\mathbf{u}]$, T by $T[\mathbf{u}]$, and M by $A[\mathbf{u}]/(\{\sum_j a_{ij}u_j\}) = \mathrm{Sym}_A M$, *we may assume that M is a cyclic A-module A/I.*

Third reduction: After localizing, completing, and killing a minimal prime of R, *we may assume that all three rings are complete and local and that R is an integral domain.*

Fourth reduction: We need the following theorem

Theorem 12. *[AFH94] If $A \to T$ is a local homomorphism of complete Noetherian local rings, and A is regular, then there is a complete regular local ring A' and a factorization*

$$\begin{tikzcd} & A' \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & T \end{tikzcd}$$

where the maps are local homomorphisms and A' is faithfully flat over A.

Replacing A and R by A' and $R \otimes_A A'$ respectively, *we may assume that the map from A to T is a surjection, with kernel a prime ideal P.* Note also that P must be generated by part of a system of parameters; say $P = (y_1, \ldots, y_h)$, where the y's are part of a s.o.p. and $h = \operatorname{ht} P$.

We will also need the following fact about regular local rings, which holds since $\otimes_A E$ preserves injectivity, where E is the injective hull of the residue field of A: If $Aw \hookrightarrow W$ is a ring map, then Aw splits from W if and only if for all ideals $I \subseteq A$, $IW \cap Aw = Iw$.

Let $Q = \ker(R \twoheadrightarrow T)$. Then $P = Q \cap A$ and $R = A + Q$, so that $T = R/Q = A/P$. Consider the map

$$\mathrm{Tor}_1^A(A/I, R) \to \mathrm{Tor}_1^A(A/I, T).$$

A typical relation is of the form

$$i_1(a_1 + q_1) + \cdots + i_k(a_k + q_k) = 0,$$

where $a_j \in A$, $q_j \in Q$, and $i_j \in I$. In particular,

$$\sum i_\nu a_\nu = -\sum i_\nu q_\nu.$$

Thus, elements of $\mathrm{Tor}_1^A(A/I, R)$ come from elements of $I \cap IQ$.[3] This says that the relation $(\overline{a_1}, \ldots, \overline{a_k})$ in A/P on $i_1, \ldots, i_k$ comes from a relation over A. This says (*Exercise*) that

$$I \cap IQ = IP$$

[3]Contrary to appearances, $IQ \nsubseteq I$.

for all ideals $I \subseteq A$.

Fifth reduction:[4] Replace A by the extended rees ring of P: $A[y_1t, \dots, y_ht, v]$, where $v = 1/t$, replace R by $R' = R[Pt, v]$, Q by a prime ideal $Q' \subseteq R'$ containing v and lying over Q, and replace P by vA'. Then localize and complete again, so that *we may assume that* $T = A/xA$*,* $P = xA$*,* $Q \subseteq R$ *lies over* xA*, and* $R = A + Q$*.*

What we need to show is that $IQ \cap I = xI$ for all ideals $I \subseteq A$. Moreover, since both the conjectures at hand imply the Direct Summand Conjecture, we may assume the Direct Summand Conjecture. Hence, *we may assume that* $IR \cap A = I$ *for all ideals* $I \subseteq A$*.*

Claim: $IQ \cap I = IQ \cap P$ for all ideals $I \subseteq A$.

Then the Vanishing Conjecture for Maps of Tor holds $\Leftrightarrow$ $IQ \cap Ax = xI$ for all $I \Leftrightarrow Ax$ splits from Q as A-modules.

This last condition is the Strong Direct Summand Conjecture, so we're done.

9 Tight closure in characteristic $p > 0$

Let $G = R^{\oplus h}$ be a free R-module, where R is a ring of characteristic $p > 0$. For $(r_1, \dots, r_h) \in G$ and $q = p^e$, set

$$(r_1, \dots, r_h)^q := (r_1^q, \dots, r_h^q) \in G.$$

For a submodule $N \subseteq G$,

$$N^{[q]} := R - \text{span of } \{n^q \mid n \in N\}.$$

If R is a domain, we write $u \in N_G^*$ if there is some $0 \neq c \in R$ such that $cu^q \in N^{[q]}$ for all powers $q > 0$ of p (equivalently, for all powers $q \gg 0$ of p).

Now, for a finitely generated R-module M, there is a surjection $G \overset{\phi}{\twoheadrightarrow} M$ where G is a finitely generated free R-module. For a submodule $N_0 \subseteq M$, let $N = \phi^{-1}(N_0) \subseteq G$. For $u \in M$, pick $u' \in G$ such that $\phi(u') = u$. We say that $u \in (N_0)_M^*$ (the *tight closure of* N_0 *as a submodule of* M) if $u' \in N_G^*$. This looks as if it depends on choices of G, ϕ, and u', but in fact it doesn't.

Recall that a ring is called *weakly* F*-regular* if every ideal is tightly closed. It's equivalent to say that for every finitely generated R-module M, every submodule of M is tightly closed. R is F*-regular* if $R_{\mathfrak{p}}$ is weakly F-regular for all $\mathfrak{p} \in \operatorname{Spec} R$.

For example, all rings of the form $k[\{X_{ij}\}]/I_t(X_{ij})$, where k is a prime characteristic field and $\{X_{ij}\}$ is a matrix of indeterminates, are F-regular.

Recall the

Theorem 13 (Tight closure Briançon-Skoda theorem). *For any equicharacteristic Noetherian ring* R*,*

$$\overline{(f_1, \dots, f_n)^n} \subseteq (f_1, \dots, f_n)^*$$

whenever $f_1, \dots, f_n \in R$*, where the bar notation stands for integral closure.*

[4]From the geometers, we learn: "When in doubt, blow up."

The following theorem was proved by Ein, Lazarsfeld and Smith in equal characteristic 0, and then by Hochster and Huneke (using tight closure methods) in characteristic $p > 0$. It is open in mixed characteristic.

Theorem 14. *If R is an equicharacteristic local ring and P is a prime ideal of finite projective dimension, then*

$$P^{(hn)} \subseteq P^n,$$

where $h = \operatorname{ht} P$.

More relevant to the homological conjectures, we have:

Theorem 15 (Phantom acyclicity criterion). *Let R be a Noetherian domain of characteristic $p > 0$, and let*

$$G_. : 0 \to R^{b_d} \xrightarrow{\alpha_d} \cdots \xrightarrow{\alpha_1} R^{b_0} \to 0$$

be a complex of finitely generated free R-modules. Set $r_i = \operatorname{rank} \alpha_i$ for $1 \leq i \leq d$ and $r_{d+1} = 0$.

Suppose that $b_i = r_{i+1} + r_i$ and $\operatorname{ht} I_{r_i}(\alpha_i) \geq i$, $i = 1, \ldots, d$. Then $G_.$ has phantom homology at the i'th spot, i.e.

$$\ker \alpha_i \subseteq (\operatorname{im} \alpha_{i+1})^*_{R^{b_i}},$$

for $i = 1, \ldots, d$.

Proof of Vanishing Conjecture for Maps of Tor*, in characteristic $p > 0$.* Let $A \subseteq R \to T$ and the A-module M be as in the standard setup for the Vanishing Conjecture, and as above we may assume that the rings are complete and local and R is a domain. Let

$$F_. : 0 \to A^{b_d} \to \cdots \to A^{b_1} \to A^{b_0} \to 0$$

be a minimal A-free resolution of M. The ranks and heights stay the same in $F_. \otimes_A R$, since R is module-finite and torsion free over A, so we have phantom homology: cycles $\subseteq$ (boundaries)*.

When we tensor with T, the phantom homology from $F_. \otimes_A R$ will vanish, since every finitely generated T-submodule is tightly closed in its ambient module. Thus,

$$\operatorname{Tor}_i^A(M, R) = H_i(F_. \otimes_A R) \xrightarrow{0} H_i(F_. \otimes_A T) = \operatorname{Tor}_i^A(M, T).$$

$\square$

Actually, all we need for this proof to work is:

- proj.dim. $M < \infty$,
- T is weakly F-regular, and
- The map $A \to R$ "preserves heights."

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