COHEN-MACAULAY RINGS

In this hour we will talk about, or build up to talking about, Cohen-Macaulay rings. This is a class of rings that is closed under the operations of localization, completion, adjoining polynomial and power series variables, and taking certain quotients. First, we need to give some definitions.

Definition: Let A be a ring and M an A-module. An element $a \in A$ is said to be M**-regular** if $ax \neq 0$ for all $0 \neq x \in M$. In other words, a is not a zero divisor on M.

Example: $M = A = k[x]$, where k is a field. Then x is regular on A.

Definition: A sequence $a_1, \ldots, a_r$ of elements of A is an M**-sequence** (or an M**-regular sequence**) if the following two conditions hold:

(1) a_1 is M-regular, a_2 is M/a_1M-regular, etc.

(2) $M/\sum a_iM \neq 0$.

REMARK: If $a_1, a_2, \ldots, a_r$ is an M-sequence, then so is $a_1^{t_1}, \ldots, a_r^{t_r}$, for any positive integers t_i. However, just because $a_1, \ldots, a_r$ is an M-sequence does not mean that a permutation of $a_1, \ldots, a_r$ is an M-sequence. In order for any permutation of the sequence to be and M-regular sequence, we would need the ring to be Noetherian local and the module to be finite

Example: The classical example of a regular sequence is $x_1, \ldots, x_r$ in the polynomial ring $A[x_1, \ldots, x_r]$.

Non-Example: Let $A = k[x, y, z]$, where k is a field. Show that $x, y(1-x), z(1-x)$ is an A-sequence, but $y(1-x), z(1-x), x$ is NOT.

Note that $z(1-x)$ is not regular on $A/(y(1-x))$ since $z(1-x)y = zy - zxy = zy - zy$ since $y = yx$ in $A/(y(1-x))$.

Let A be a Noetherian ring and M a finite A-module. If $\mathbf{x} = x_1, \ldots, x_n$ is an M-sequence, then the chain $(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, x_2, \ldots, x_n)$ ascends strictly. Therefore, an M-sequence can be extended to a maximal such sequence, since A is Noetherian, and hence the chain must terminate.. An M-sequence $x_1, \ldots, x_n$ is maximal if $x_1, \ldots, x_{n+1}$ is NOT an M-sequence for any $x_{n+1} \in R$.

REMARK: All maximal M-sequences have the same length if M is finite (and A is Noetherian).

Let A be a local ring.

Definition: We call this length the **depth** of M. If we are talking about M-sequences in a non-maximal ideal I of A, then we use the notation $\operatorname{depth}(I, M)$.

Another, more technical definition of depth, is $\inf\{i\colon \operatorname{Ext}^i_A(k, M) \neq 0\}$, where k is the residue field of A. Likewise, $\operatorname{depth}(I, M) = \inf\{i\colon \operatorname{Ext}^i_A(A/I, M) \neq 0\}$.

Definition: Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and M a finite A-module. M is called **Cohen-Macaulay (CM)** if $M \neq 0$ and depth $M = \dim M$. If A is itself a Cohen-Macaulay module, we say that A is a **Cohen-Macaulay ring**.

QUESTION: What happens if A is not local?

Definition: A Noetherian ring A is said to be a CM ring if $A_\mathfrak{m}$ is a CM local ring for every maximal ideal $\mathfrak{m}$ of A.

Example: The rings $k[X_1, \ldots, X_n], k[[X_1, \ldots, X_n]],$, $k[X, Y, Z]/(XY - Z)$, and $k[X, Y, Z, W]/(XY - ZW)$ are all Cohen-Macaulay, for example.

Non-Example: The ring $A = k[[X^4, X^3Y, XY^3, Y^4]] \subset k[X, Y]$ is NOT Cohen-Macaulay.

Note that A has dimension 2 since $\{X^4, Y^4\}$ is an s.o.p.. In particular, $(X^3Y)^4 = X^{12}Y^4 \in (X^4, Y^4)$; likewise for XY^3. Thus, $\mathfrak{m} \subset (X^4, Y^4)$. We'll show that Y^4 is not regular on $A/(X^4)$. $Y^4(X^6Y^2) = X^6Y^6 = X^4(X^2Y^6) \in (X^4)$, but $X^6Y^2 \notin (X^4)$ since $X^6Y^2 = X^4(X^2Y^2)$, and $X^2Y^2 \notin A$. Thus, we have found an s.o.p. that is not A-regular.

In order to show that the above example was not Cohen-Macaulay, we used the following result.

THEOREM: Let $(A, \mathfrak{m})$ be a CM local ring.

(a) For a proper ideal I of A we have ht $I = \text{depth}(I, A)$ and ht $I +$ dim $A/I = \dim A$.

(b) For any sequence $a_1, \ldots, a_r \in \mathfrak{m}$ the following four conditions are equivalent:

(i) $a_1, \ldots, a_r$ is an A-sequence

(ii) $\text{ht}(a_1, \ldots, a_i) = i$ for $1 \leq i \leq r$

(iii) $\text{ht}(a_1, \ldots, a_r) = r$

(iv) $a_1, \ldots, a_r$ is part of a system of parameters of A

THEOREM: Let $(A, \mathfrak{m})$ be a Noetherian local ring and M a finite A-module.

(a) If $a_1, \ldots, a_r \in \mathfrak{m}$ is an M-sequence and we set $M' = M/(a_1, \ldots, a_r)M$, then M is a CM module $\Leftrightarrow M'$ is a CM module.

(b) If M is a CM module then $M_\mathfrak{p}$ is a CM module over $A_\mathfrak{p}$ for every $\mathfrak{p} \in \text{Spec}(A)$, and if $M_\mathfrak{p} \neq 0$, then $\text{depth}(\mathfrak{p}, M) = \text{depth}_{A_\mathfrak{p}} M_\mathfrak{p}$.

(c) Let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. Then (i) depth $A =$ depth $\hat{A}$ and (ii) A is CM $\Leftrightarrow \hat{A}$ is CM.

(d) If A is CM, then so are $A[X]$ and $A[[X]]$.

Sometimes one only needs a ring or module to be Cohen-Macaulay in "low dimension". This concept is called the Serre condition S_i, for $i \geq 0$.

<u>Definition</u>: The condition S_i on A means that depth $A_\mathfrak{p} \geq \min\{\text{ht } \mathfrak{p}, i\}$ for all $\mathfrak{p} \in \text{Spec}(A)$.

REMARK 1: Of course, S_0 always holds; S_1 says that all the associated primes of A are minimal, that is A does not have embedded associated primes; for an integral domain, S_2 is equivalent to the condition that every prime divisor of a non-zero principal ideal has height 1

REMARK 2: (S_i) for all $i \geq 0$ is just the definition for CM ring.

REMARK 3: If S_i holds on A, then S_j also holds, for all $j < i$.

Definition: To say that a finite module M satisfies S_i means that depth $M_{\mathfrak{p}} \geq \min\{\dim\ M_{\mathfrak{p}}, i\}$, for all $\mathfrak{p} \in \mathrm{Spec}(A)$.

Definition: We can also define the **Serre condition** R_i, for $i \geq 0$. A Noetherian ring A satisfies R_i if $A_{\mathfrak{p}}$ is a regular local ring for all prime ideals $\mathfrak{p}$ in A of height less than or equal to i.

REMARK 1: If R_i holds on A, then R_j also holds, for all $j < i$, since $A_{\mathfrak{p}} \cong (A_{\mathfrak{q}})_{\mathfrak{p}A_{\mathfrak{q}}}$, for $\mathfrak{p} \subset \mathfrak{q}$.

REMARK 2: Recall that the a Noetherian ring A is normal if and only if the conditions S_2 and R_1 are satisfied.

Recall that, in general, depth $M \leq \dim\ M \leq \dim\ A$, for a finite A-module M.

Definition: A **small CM module** is a finitely-generated module M such that depth $M = \dim\ R$. We can also define a **big CM module**. Let $x_1, \ldots, x_d$ be a system of parameters. If M is a (not necessarily finitely-generated) A -module such that $(x_1, \ldots, x_d)M \neq M$ and $x_1, \ldots, x_d$ is an M-regular sequence, then M is called a big CM module.

SMALL COHEN-MACAULAY CONJECTURE: If R is a complete local Noetherian ring, then R has a small CM module.

BIG COHEN-MACAULAY CONJECTURE: If R is a local Noetherian ring, then R has a big CM module.

This class of Gorenstein rings is also closed under localization, completion, and adjoining polynomial and power series variables. We start with some definitions.

Definition: An A-module I is called **injective** if given any diagram of A-module homomorphisms:

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{g} & L \\ & & \big\downarrow{\scriptstyle f} & & \\ & & I & & \end{array}$$

with top row exact, there exists an A-module homomorphism $h : L \to I$ such that $h \circ g = f$.

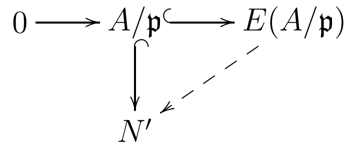
Definition: Let A be a Noetherian ring and M and N (not necessarily finitely-generated) A-modules. Then $M \overset{\alpha}{\hookrightarrow} N$ is called **essential** (i.e., N is an **essential extension** of M if $N_0 \cap M \neq 0$ for every non-zero submodule N_0 of N.

Definition: An **injective hull** of M is an injective module $E \supset M$ such that $M \hookrightarrow E$ is essential. The notation is $E(M)$.

THEOREM: *Every injective module over a Noetherian ring is a direct sum of indecomposable injective modules.*

PROOF

Say that a family $\mathfrak{F} = \{E_\lambda\}$ of indecomposable (meaning the E_λ can not be written as the direct sum of two submodules) injective submodules of M is **free** if the sum in M of the E_λ's is direct; i.e., if, for any finite number $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_n}$ of them, $E_{\lambda_1} \cap (E_{\lambda_2} + \cdots + E_{\lambda_n}) = 0$. Let $\mathfrak{M}$ be the set of all free families $\mathfrak{F}$, ordered by inclusion. By Zorn's Lemma, $\mathfrak{M}$ has a maximal element, say $\mathfrak{F}_0$. Write $N = \sum_{E \in \mathfrak{F}_0} E$. Then N is injective, since any direct sum of injective modules is injective. Since an injective submodule is always a direct summand, we have $M = N \oplus N'$. If $N' \neq 0$, then since it's a direct summand of the injective module M it must be injective itself. Let $\mathfrak{p} \in \operatorname{Ass}(N')$. Then we have the following diagram:



where $L \to N'$ is an injection since the other two maps are injections. Thus, N' contains a direct summand E' isomorphic to $E(A/\mathfrak{p})$, which is indecomposable. Since $M = N \oplus N'$, this means that $E' \cap N = 0$; i.e., $\mathfrak{F}_0 \cup \{E'\}$ is a free family, contradicting the maximality of $\mathfrak{F}_0$. Hence $N' = 0$, and $M = N$.

Definition: An A-module M has **injective dimension** $\leq n$ **(inj dim** $M \leq n$**)** if there is an injective resolution

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$$

.

If no such finite resolution exists, then inj dim (M) is defined to be ∞.

Definition: A Noetherian local ring A is a **Gorenstein ring** if any of the following equivalent conditions hold:

(1) inj $\dim(A) < \infty$

(1') inj $\dim(A) = n$

(2) $\mathrm{Ext}^i_A(k, A) = 0$ for $i \neq n$ and $\cong k$ for $i = n$

(3) $\mathrm{Ext}^i_A(k, A) = 0$ for some $i > n$

(4) $\mathrm{Ext}^i_A(k, A) = 0$ for $i < n$ and $\cong k$ for $i = n$

(4') A is a CM ring and $\mathrm{Ext}^n_A(k, A) \cong k$

(5) A is a CM ring, and every parameter ideal (i.e., generated by a system of parameters) of A is irreducible (meaning that if $I = J \cap J'$, then either $I = J$ or $I = J'$)

(5') A is a CM ring and there exists an irreducible parameter ideal

REMARK: There is another useful characterization of Gorenstein rings. Namely, A is a CM ring of type 1

Definition: The **type** of a non-zero finite A-module M of depth t is $r(M) =$

$\dim_k \operatorname{Ext}^t_A(k, M)$.

Example: The rings $k[X_1, \ldots, X_n]$, $k[[X_1, \ldots, X_n]]$, $k[X, Y, Z]/(XY - Z)$, and $k[X, Y, Z, W]/(XY - ZW)$ are Gorenstein rings.

Non-Example: (Gor $\subset$ CM) Let $A = k[X, Y]/(X^2, XY, Y^2)$, which has dimension 0, and hence is a CM ring. The depth of A is also zero, so the type of A is $\dim_k \operatorname{Hom}_A(k, A)$. Now $\operatorname{Soc}(A) = (0 : \mathfrak{m})_A \cong \operatorname{Hom}_A(k, A) = \mathfrak{m}$ in this case. Thus, $\dim_k \operatorname{Hom}_A(k, A) = \dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$. Therefore, since type of A is greater than 1, A can not be Gorenstein.

Definition:A Noetherian ring is a **Gorenstein ring** if its localization at every maximal ideal is a Gorenstein local ring.

THEOREM: Let A be a Noetherian ring.

(a) Suppose A is Gorenstein. Then for every multiplicatively closed set S in A the localized ring A_S is also Gorenstein. In particular, $A_\mathfrak{p}$ is Gorenstein for every $\mathfrak{p} \in \operatorname{Spec}(A)$.

(b) Suppose $\mathbf{x}$ is an A-regular sequence. If A is Gorenstein, then so is $A/(\mathbf{x})$.

(c) Suppose A is local. Then A is Gorenstein if and only if its completion $\hat{A}$ is Gorenstein.

(d) If A is Gorenstein, then so are $A[X]$ and $A[[X]]$.

REMARK: One way to check if a ring is Gorenstein is to first check if it is Cohen-Macaulay. If so, then find a system of parameters, kill it, and then compute the Socle; determine how many linearly independent elements are in the Socle. This is the type.