AUSLANDER-BUCHSBAUM FORMULA

This formula is an "effective instrument for the computation of the depth of a module", according to Bruns-Herzog, Cohen-Macaulay Rings.

THEOREM: (Auslander-Buchsbaum) *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and $M \neq 0$ a finite A-module. If proj dim $M < \infty$, then*

$$proj\ dim M + depth M = depth A.$$

Recall the following definitions:

<u>Definition</u>: An A-module P is called **projective** if given any diagram of A-module homomorphisms:

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle f} & & \\ N & \xrightarrow{g} & L & \longrightarrow & 0 \end{array}$$

with bottom row exact, there exists an A-module homomorphism $h : P \to N$ such that $g \circ h = f$.

NOTE: Recall that over a Noetherian local ring, flat = projective = free.

<u>Definition</u>: An A-module M has **projective dimension** $\leq n$ **(**$pd(M) \leq n$**)** if there is a projective resolution

$$0 \to P_n \to P_1 \to \cdots \to P_0 \to M \to 0$$

.

If no such finite resolution exists, then $pd(M)$ is defined to be ∞; otherwise, if n is the least such integer, define $pd(M) = n$.

We need some preliminary definitions and results before proving the Auslander-Buchsbaum formula.

<u>Definition</u>: Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring and let M be a finite A-module. An complex $L_. : \ldots L_i \stackrel{d_i}{\to} L_{i-1} \stackrel{d_{i-1}}{\to} \cdots \stackrel{d_2}{\to} L_1 \stackrel{d_1}{\to} L_0 \stackrel{\epsilon}{\to} M \to 0$ is called a **minimal free resolution** of M if it satisfies the three conditions

(1) each L_i is a finite free A-module, (2) in the complex $L_. \otimes k$, $\overline{d}_i = 0$, or in other words, $d_i L_i \subset \mathfrak{m} L_{i-1}$ for all i, and (3) $\overline{\epsilon} : L_0 \otimes k \to M \otimes k$ is an isomorphism.

REMARK: Note that a minimal free resolution of a finite A-module M can be constructed as follows: Let $x_1, \ldots, x_{\beta_0}$ be a minimal system of generators of M. Define $\phi_0 : A^{\beta_0} \to M$ by $\phi_0(e_i) = x_i$, where $e_1, \ldots, e_{\beta_0}$ is the canonical basis of A^{β_0}. Let β_1 be the number of minimal generators of $\mathrm{Ker}(\phi_0)$. In a similar way, define an epimorphism $A^{\beta_1} \to \mathrm{Ker}(\phi_0)$. Then map $\phi_1 : A^{\beta_1} \to A^{\beta_0}$ via the composition $A^{\beta_1} \to \mathrm{Ker}(\phi_0) \to A^{\beta_0}$. Continue in this way.

NOTE 1: Any two minimal free resolutions of M are isomorphic as complexes (which means that there is a chain map between the two complexes which is degree-wise an isomorphism)

NOTE 2: The number β_i is called the ***i*-th Betti number.**

PROPOSITION: *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finite A-module. Then*

$$pd(M) = sup\{i : Tor_i^A(k, M) \neq 0\}.$$

REMARK: Of course, if $pd(M) = n$, then $\mathrm{Tor}_i^A(N, M) = 0$ for all $i > n$ and for any A-module N; i.e., $\mathrm{Tor}_i^A(-, M)$ vanishes for $i > n$. However, in order to conclude that $pd(M) = n$, it suffices to show that these Tor's vanish when $N = k$.

PROOF

Recall that $pd(M)$ is the minimum of lengths of projective resolutions of M. Also, recall that $\mathrm{Tor}_i^A(k, M) = \mathrm{H}_i(k \otimes P_.)$, where $P_.$ is a projective resolution of M. Suppose $pd(M) = n$. Because the definition of Tor is independent of the projective resolution chosen, we may assume that the length of $P_.$ is n. Then $\mathrm{H}_i(k \otimes P_.) = 0$ for $i > n$ since the complex $P_.$ has only zeroes after the n-th place; i.e., $\mathrm{Tor}_i^A(k, M) = 0$ for $i > n$. Thus, $pd(M) \geq sup\{i : \mathrm{Tor}_i^A(k, M) \neq 0\}$. We need to show that $\mathrm{Tor}_n^A(k, M) \neq 0$.

Suppose that $F_. : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$ is a free resolution of

M. If $F_.$ is minimal, then the maps $\overline{d}_i$ in the complex $k \otimes F_.$ are all zero, by definition. Thus,

$$\mathrm{Tor}_i^A(k,M) = \ker(\overline{d}_i)/\mathrm{im}(\overline{d}_{i+1}) = k \otimes F_i$$

since $\overline{d}_i = 0 \Rightarrow \ker(\overline{d}_i) = k \otimes F_i$ and $\overline{d}_{i+1} = 0 \Rightarrow \mathrm{im}(\overline{d}_{i+1}) = 0$. But k and F_i finitely-generated $\Rightarrow k \otimes F_i = 0 \Leftrightarrow F_i = 0$. Since $F_n \neq 0$, $k \otimes F_n \neq 0$, and hence $\mathrm{Tor}_n^A(k,M) \neq 0$. Thus, $pd(M) = sup\{i : \mathrm{Tor}_i^A(k,M) \neq 0\}$.

PROPOSITION: *Let $(A, \mathfrak{m})$ be a Noetherian local ring, and M a finite A-module. If $x \in \mathfrak{m}$ is A-regular and M-regular, then $pd_A(M) = pd_{A/(x)}(M/xM)$.*

PROOF

Choose an augmented minimal free resolution $F_.$ of M. Since x is A- and M-regular, $F_. \otimes A/(x)$ is exact. Therefore, it is a free resolution of M/xM over $A/(x)$. Recall that $\mathrm{Tor}_i^{A/(x)}(k, M/xM) = \mathrm{Tor}_i^A(k,M)$ for all $i \geq 0$. (The requirements for this are that x is A- and M-regular and that x kills k.) Therefore $pd_{A/(x)}(M/xM) = sup\{i : \mathrm{Tor}_i^{A/(x)}(k, M/xM) = sup\{i : \mathrm{Tor}_i^A(k,M) \neq 0\} = pd_A(M)$.

PROOF OF AUSLANDER-BUCHSBAUM FORMULA

IDEA: Induct on depth(A).

By hypothesis, $pd(M)$ is finite; say $pd(M) = n$. Thus, M has a minimal free resolution:

$$F_. : 0 \to F_i \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \to 0$$

Suppose depth $A = 0$. Then $\mathfrak{m} \in \mathrm{Ass}(A) \Rightarrow$ there exists a short exact sequence $0 \to A/\mathfrak{m} \to A \to C \to 0$. From this we get a long exact sequence:

Now $\mathrm{Tor}_i^A(k,M)=0, \forall i \geq n$, so in particular, $\mathrm{Tor}_{n+1}^A(C,M) \cong \mathrm{Tor}_n^A(k,M) \neq 0$. This is a contradiction unless $n=0$, since $\mathrm{Tor}_{n+1}^A(-,M)$ vanishes beyond n. Thus, $\mathrm{Tor}_1^A(k,M)=0 \Rightarrow M$ is projective, which means that M is free over A, since projective = free over a Noetherian local ring. Thus,

$$0+\mathrm{depth}(M)=\mathrm{depth}(A),$$

showing the formula holds in this case.

Next, let depth $A>0$. If depth $M>0$, then $\mathfrak{m} \notin \mathrm{Ass}(A)$ and $\mathfrak{m} \notin \mathrm{Ass}(M)$. Therefore, we can find an $x \in \mathfrak{m}$ such that x is non-zero divisor on both A and M. Then $\mathrm{depth}_{A/(x)}(A/(x)) =$ depth $A-1$, $\mathrm{depth}_{A/(x)}(M/xM) =$ depth $M-1$, and $pd_{A/(x)}(M/xM)=pd_A(M)$. Therefore, by induction on depth A, $pd_{A/(x)}(M/xM)+$ $\mathrm{depth}_{A/(x)}(M/xM) = \mathrm{depth}_{A/(x)}(A/(x))$, and consequently, $pd_A(M)+$ $\mathrm{depth}_A(M)=\mathrm{depth}_A(A)$. Therefore, we need only consider the case depth $M=0$. Take the short exact sequence

$$0 \to K \to A^t \to M \to 0.$$

Then $pd_A(K)=pd_A(M)-1$ and depth $(K)=1$. (This follows from the fact that depth $K \geq \min\{\text{depth } A^t, \text{ depth } M+1\}=1$ and $0=$ depth $M \geq \min\{\text{depth } K-1, \text{ depth } A^t\ \} =$ depth $K-1 \Rightarrow$ depth $K \leq 1$; thus, depth $K=1$.) We have proven above the case where depth $K>0$. Thus, $pd_A(K)+$ $\mathrm{depth}_A(K)=$ depth $A \Rightarrow pd_A(M)-1+$ $\mathrm{depth}_A(M)+1=$ depth A.

Example: A free module F is projective, so $pd(F)=0$. By the Auslander-Buchsbaum formula, depth $F=$ depth A.

Example: Consider a Noetherian local ring $(A,\mathfrak{m},k)$. Let x be an A-regular element. The short exact sequence

$$0 \to A \xrightarrow{\cdot x} A \to A/(x) \to 0$$

shows that $pd(A/(x))=1$. Thus, depth $A/(x)=$ depth $A-1$.

Example: Consider the ring $A = k[[X_1, \dots, X_n]]$. Then $A/(X_1, \dots, X_i)$ has depth $n - i$; therefore, $pd(A/(X_1, \dots, X_i)) = i$. (This is a way to construct rings with projective dimension n, for any $n \in \mathbb{N}$.

Example: Let $A = k[[X, Y]]/(X^2, XY)$. Then $\mathfrak{m} = (x, y)$ is annihilator of x, so $\mathfrak{m} \in \mathrm{Ass}(A)$. Consequently, $A/\mathfrak{m} \hookrightarrow A$, which implies that $\mathrm{Hom}_A(k, A) \neq 0$, or $\inf\{i\colon \mathrm{Ext}^i_A(k, A) \neq 0\} = 0$; i.e., depth $A = 0$. Set $M = A/(x) \cong k[[Y]]$. Then M is a regular local ring of dimension one. Thus, depth $M = 1$. By the formula, we see that M can not have finite projective dimension.

Next is an interlude on regular local rings. Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring.

Definition: Recall that a **system of parameters** of A is a sequence of elements $a_1, \dots, a_r \in \mathfrak{m}$ which generate an $\mathfrak{m}$-primary ideal. If the elements generate $\mathfrak{m}$ itself, then $a_1, \dots, a_r$ are called a **regular system of parameters**.

Definition: A **regular local ring** is one in which the maximal ideal is generated by a regular system of parameters.

REMARK 1: Recall that a regular local ring is always a domain.

REAMRK 2: As with CM and Gorenstein rings, if a ring is not local, then to say it is regular means that the localization at every prime is a regular local ring.

THEOREM: (Auslander-Buchsbaum-Serr) *The following conditions are equivalent for a Noetherian local ring A:*

- **(a)** *A is regular*
- **(b)** *all f.g. A-modules have finite projective dimension*
- **(c)** *the residue field, k, of A has finite projective dimension*

As with Cohen-Macaulay and Gorenstein rings, the class of regular rings is closed under the usual operations:

THEOREM: (Serre) *Let A be a regular local ring and $\mathfrak{P}$ a prime ideal. Then $A_{\mathfrak{P}}$ is again regular.*

THEOREM: *Let A be a Noetherian local ring. Then*

(a) *A if regular $\Leftrightarrow$ $\hat{A}$ is regular*

(b) *If A is regular, then R/I is regular $Leftrightarrow I$ is an ideal generated by a subset of a regular system of parameters.*

(c) *A regular implies that $A[X_1, \ldots, X_n]$ and $A[[X_1, \ldots, X_n]]$ are regular.*

Definition: A Noetherian local ring A is a **complete intersection (or c.i.)** if the completion $\hat{A}$ is a quotient of a complete regular local ring R by an ideal generated by an R-sequence.

REMARK: regular $\subset$ c.i. $\subset$ Gorenstein $\subset$ CM

Example: (Matsumura Exercise 21.3) If k is field, then $A = k[[X, Y, Z]]/(x^2 - Y^2, Y^2 - Z^2, XY, YZ, XZ)$ is Gorenstein but not a complete intersection.

Exercise: (Suggested by Jan) Let $R = \mathbb{C}[[X, Y, Z]]/(X^2, Y^2, Z^2)$. Find all ideals I such that R/I is Gorenstein, but not a complete intersection.

Exercise: (Suggested by Paul) Show how to compute depth using the Auslander-Buchsbaum formula.