STIFFNESS: A TALK GIVEN AT A MINICOURSE ON HOMOLOGICAL CONJECTURES IN COMMUTATIVE ALGEBRA, AT THE UNIVERSITY OF UTAH

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This talk is a summary of some of the material in the papers [SS04] and [SS03].

1. Stiffness

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring. Let $\phi : A^m \to A^n$ be a homomorphism of free modules, represented by the $n \times m$ matrix M. For $r \geq 1$, $I_r(\phi) :=$ the ideal in A generated by the $r \times r$ minors of M. [1]

For a bounded complex of finitely generated free modules:

$$F_. = (0 \to F_s \xrightarrow{d_s} F_{s-1} \xrightarrow{d_{s-1}} \cdots \xrightarrow{d_1} F_0)$$

let $f_i := \operatorname{rank} F_i$, and $r_i := \sum_{j=i}^{s} (-1)^{j-i} f_j$.

The Buchsbaum-Eisenbud Acyclicity Criterion [BE73] says that $F_.$ is exact if and only if $\operatorname{gr}(I_{r_i}(d_i)) \geq i$ and $\operatorname{rank} d_i = r_i$ for $i = 1, \ldots, s$.

We say that the complex is *minimal* if $\operatorname{im} d_i \subseteq \mathfrak{m} F_{i-1}$ for $i = 1, \ldots, s$.

For a map $g : F \to F'$ of finitely generate free modules, we will call an ideal $\mathcal{C}$ of A a *column ideal belonging to* g if for some choice of bases for F and F', $\mathcal{C}$ is generated by the entries of one of the columns of the matrix representing g.

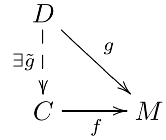
Definition 1.1. Let $F_. = (0 \to F_s \xrightarrow{d_s} F_{s-1} \xrightarrow{d_{s-1}} \cdots \xrightarrow{d_1} F_0)$ be an exact and minimal complex of finitely generated free A-modules. We say that $F_.$ is *stiff* if $\operatorname{gr} \mathcal{C} \geq i$ for every column ideal belonging to d_i, $i = 1, \ldots, s$.

The ring A is called *stiff* if every such complex over it is stiff.

2. Auslander-Buchweitz theory

[2]

Definition 2.1. Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring of dimension d, let $f : C \to M$ be a map of finitely generated R-modules, where C is a maximal Cohen-Macaulay ('mCM') module. f is called an *mCM-precover* of M if for every mCM module D, the map $\operatorname{Hom}(D, C) \xrightarrow{f_*} \operatorname{Hom}(D, M)$ is surjective:



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[1]It is standard that this definition is independent of choice of bases for the free modules involved.

[2]This has been pushed further. [Has00]

Such an f is called an *mCM-cover* if moreover it is left-minimum (i.e. whenever $g : C \to C$ is an endomorphism with $f \circ g = f$, it follows that g is an automorphism).

Dually, if $h : M \to I$ is a map of finitely generated R-modules, where I is a module of finite injective dimension ('fid'), we say that h is an *fid-preenvelope* if for every fid module J, the map $\mathrm{Hom}(I, J) \xrightarrow{f^*} \mathrm{Hom}(M, J)$ is surjective:

$$\begin{array}{ccc} M & \xrightarrow{h} & I \\ & \underset{g}{\searrow} & \Big\downarrow{\scriptstyle \exists \tilde{g}} \\ & & J \end{array}$$

Such an h is called an *fid-envelope* if moreover it is right-minimum.

Theorem 2.2 (Auslander-Buchweitz [AB89])**.** *Let R be as above. Then mCM-covers f and fid-envelopes h exist and are unique up to isomorphism. Moreover, f is surjective, h is injective,* $\ker f$ *is fid, and* $\operatorname{coker} h$ *is mCM.*

Definition 2.3. For a finitely generated R-module M, the *free rank of M*, denoted $\text{f-rank}\, M$, is the integer n where $M \cong M' \oplus R^n$, and M' has no nonzero free direct summand.

For an mCM-cover $C \twoheadrightarrow M$, define $\delta(M) := \text{f-rank}\, C$, Auslander's *delta-invariant* of M. (By the Theorem, this is in fact an invariant of M.)

Note that $\delta(M) = \inf\{\text{f-rank}\, D \mid D \twoheadrightarrow M \text{ and } D \text{ is mCM.}\}$.

The surjection $R^t \twoheadrightarrow M$ induces a map

$$\mathrm{Ext}^d(k, R^t) \xrightarrow{\sigma} \mathrm{Ext}^d(k, M)$$

Then we have $\delta(M) = \dim_k(\operatorname{im} \sigma)$.

Definition 2.4. Consider the natural map

$$\mathrm{Ext}^j(k, M) \xrightarrow{i} H^i_{\mathfrak{m}}(M).$$

Then $\dim_k \mathrm{Ext}^j(k, M)$ is the j'th Bass number of M, and we call $\nu^j(M) := \dim_k(\operatorname{im} i)$ the j'th *reduced Bass number* of M.

Proposition 2.5. *If $M \hookrightarrow I$ is an fid-envelope of M, then* $\text{f-rank}\, I = \nu^d(M)$*.*

Lemma 2.6. $\nu^d(M) > 0 \Leftrightarrow$ *there is a map $M \to R$ such that the induced map* $\mathrm{Ext}^d(k, M) \to \mathrm{Ext}^d(k, R)$ *is nonzero.*

This leads us to the relationship with the Canonical Element Conjecture ('CEC'):

Suppose that A has a canonical module K_A. Then A satisfies CEC $\Leftrightarrow \nu^d(K_A) > 0$.

Definition 2.7. For a Noetherian local ring $(A, \mathfrak{m}, k)$, we say that a finitely generated A-module K is a *canonical module* of A if $K^\vee \cong H^d_{\mathfrak{m}}(A)$. Here $(-)^\vee$ represents the Matlis dual of a module.

3. Taking advantage

If we want to prove the Canonical Element Conjecture for a Noetherian local ring A, it is enough to prove it for $\hat{A}$, so we may assume that A is complete. Then by the Cohen Structure Theorem, $A \cong R/\mathfrak{a}$, where R is a complete intersection (hence a Gorenstein ring) and $\mathfrak{a}$ is an ideal consisting of zerodivisors of R.

Now, let $\mathfrak{b} = 0 :_R \mathfrak{a}$. Then $0 \neq \mathfrak{b} \cong \mathrm{Hom}_R(A, R)$, so that in particular, $\mathfrak{b}$ *is an* A*-module. In fact, is the canonical module of* A! Remember: A satisfies CEC $\Leftrightarrow \nu^d_A(\mathfrak{b}) > 0$.

The following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ext}^d_A(k, \mathfrak{b}) & \longrightarrow & H^d_{\mathfrak{m}_A}(\mathfrak{b}) \\ \downarrow & & \| \\ \mathrm{Ext}^d_R(k, \mathfrak{b}) & \longrightarrow & H^d_{\mathfrak{m}_R}(\mathfrak{b}) \end{array}$$

The top map is zero whenever the bottom one is. So, A satisfies CEC $\Rightarrow \nu^d_R(\mathfrak{b}) > 0$. From the short exact sequence of R-modules

$$0 \to \mathfrak{b} \to R \to R/\mathfrak{b} \to 0,$$

we get a long exact sequence of Ext-modules, part of which looks like

$$\mathrm{Ext}^d_R(k, \mathfrak{b}) \xrightarrow{f} \mathrm{Ext}^d_R(k, R) \xrightarrow{g} \mathrm{Ext}^d_R(k, R/\mathfrak{b}),$$

and since R is Gorenstein, the middle of these three k-vector spaces is isomorphic to k. Hence, we have:

$$\nu^d_R(\mathfrak{b}) > 0 \iff f \text{ is onto} \iff g = 0 \iff \delta_R(R/\mathfrak{b}) = 0.$$

This statement about R implies that A satisfies a dual form of the Monomial Conjecture ('MC'), which was shown to be true by J. Stückrad. Hochster [Hoc83] showed that for any fixed positive integer d, every A of dimension d with residual characteristic $p > 0$ satisfies CEC $\iff$ every such ring satisfies MC.

Our philosophy is that these homological conjectures are really about Gorenstein rings, their ideal theory and their module theory.

Another statement equivalent to the Monomial conjecture is that $\delta_R(R/\mathfrak{b}) = 0$ for every unmixed ideal $\mathfrak{b}$ of zerodivisors of R, where R is a Gorenstein local ring.

Question: Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of the Gorenstein local ring R. To prove the monomial conjecture, is it enough to show that $\delta_R(R/\mathfrak{p}_j) = 0$ for all j?

4. Tying up

Stiffness essentially is about first syzygies:

Definition 4.1. A class Γ of rings is called *consistent* if whenever $A \in \Gamma$ and $x \in A$ is a nonzerodivisor, then $A/(x) \in \Gamma$.

Theorem 4.2. *For a consistent class* Γ *of rings, every* $A \in \Gamma$ *is stiff if and only if for every minimal bounded free complex* $(F_., d_.)$ *as in Section 1 over every such ring,* $\mathrm{gr}\,\mathcal{C} \geq 1$ *for every column ideal* $\mathcal{C}$ *belonging to* d_1*, regardless of bases.*

Proposition 4.3. *Let* R *be a Gorenstein local ring. Then* $\delta_R(R/\mathfrak{b}) = 0$ *for all unmixed nonzero ideals* $\mathfrak{b}$ *consisting of zerodivisors* $\iff$ *every minimal generator of a syzygy* Z *of finite projective dimension has annihilator 0.*

Remarks about syzygies: Let $(R, \mathfrak{m})$ be Gorenstein. Then M is a syzygy $\iff M$ is torsionless (i.e. $M \hookrightarrow M^{**}$ is injective) $\iff \mathrm{Ass}M \subseteq \mathrm{Ass}R$.

For a cyclic module $M \cong R/\mathfrak{b}$, $\mathrm{Ass}M \subseteq \mathrm{Ass}R \iff \mathfrak{b}$ is unmixed and consists of zerodivisors $\iff \mathfrak{b} = 0 :_R \mathfrak{a}$ for some ideal $ia \subseteq R$ consisting of zerodivisors.

Proof of the proposition. ($\Rightarrow$): uses Auslander-Buchweitz.

($\Leftarrow$): Consider the following short exact sequence arising from the fid-envelope of $R/\mathfrak{b}$:

$$0 \to R/\mathfrak{b} \to Z \to D \to 0.$$

Here Z has finite injective dimension and D is a maximal Cohen-Macaulay module. We have:

$$\mathrm{Ass}_R Z \subseteq \mathrm{Ass}_R R/\mathfrak{b} \cup \mathrm{Ass}_R D \subseteq \mathrm{Ass} R.$$

The first containment follows from the short exact sequence. The second containment follows because $\mathfrak{b}$ is unmixed and D is maximal Cohen-Macaulay.

Hence, $R/\mathfrak{b}$ maps into $\mathfrak{m}Z$ in the short exact sequence above, so $\delta_R(R/\mathfrak{b}) = 0$. □

Conclusion: Every Noetherian local ring of dimension d and residual characteristic $p > 0$ satisfies CEC $\iff$ every Gorenstein ring of this type is stiff.

(Note: Using big Cohen-Macaulay modules, one can show that every Gorenstein ring of equal characteristic is stiff.)

5. Questions

(1) Is stiffness independent of (finite) resolution?
(2) Does stiffness incorporate rank criteria, as in Buchsbaum-Eisenbud?
(3) What about the minors of matrices?

References

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