Global bifurcation in variational inequalities

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1 Examples

A unilateral problem

$$-u'' = \lambda(u + u^3),\ t \in (0, \pi),$$

subject to the unilateral constraints

$$\begin{cases} 0 \le u(0),\ 0 \le u(\pi) \\ u'(0) \le 0 \le u'(\pi) \\ u(0)u'(0) = 0 = u(\pi)u'(\pi). \end{cases}$$

Dirichlet boundary conditions:

$$u(0) = 0 = u(\pi),$$

where, however λ must be restricted so that the second of the unilateral conditions hold, i.e.

$$u'(0) < 0 < u'(\pi).$$

Thus, for example, the problem may not have any solutions u, with $u(t) > 0,\ t \in (0, \pi)$, nor any solutions u with $u(t) > 0$ for t in a neighborhood of 0 and $u(t) < 0$ for t in a neighborhood of π.

Bifurcation points

$$\lambda = n^2,\ n = 1, 3, \cdots$$

Neumann boundary conditions:

$$u'(0) = 0 = u'(\pi),$$

where, however λ must be restricted so that the first of the unilateral conditions hold, i.e.

$$u(0),\ u(\pi) > 0.$$

Bifurcation points

$$\lambda = n^2,\ n = 0, 2, 4, \cdots$$

Mixed Dirichlet and Neumann boundary conditions:

$$u(0) = 0 = u'(\pi),$$

where λ must be restricted so that the first and the second of the unilateral conditions hold, i.e.

$$u'(0) < 0,\ u(\pi) > 0.$$

Bifurcation points

$$\lambda = \left(\frac{2n-1}{2}\right)^2,\ n = 2, 4, \cdots$$

Mixed Neumann and Dirichlet boundary conditions:

$$u'(0) = 0 = u(\pi),$$

where now λ must be restricted so that the first and the second of the unilateral conditions hold, i.e.

$$u(0) > 0,\ u'(\pi) > 0.$$

In this case we obtain the set of bifurcation points as for the other set of mixed boundary conditions considered above.

1.1 An equivalent variational inequality

Let K be defined by

$$K = \{u \in H^1(0,\pi) : u(0) \geq 0,\ u(\pi) \geq 0\}.$$

Then above unilateral problem is equivalent to the variational inequality

$$\begin{cases} \int_0^\pi u'(v-u)' - \lambda(u+u^3)(v-u) \geq 0,\ u \in K, \\ \forall v \in K \end{cases}$$

Let I_K be the indicator function of the set K, i.e.

$$I_K(u) = \begin{cases} 0, & u \in K \\ \infty, & u \notin K, \end{cases}$$

The variational inequality is equivalent to the variational inequality

$$
\begin{cases}
\displaystyle\int_0^\pi u'(v-u)' - \lambda(u+u^3)(v-u) \\
+I_K(v) - I_K(u) \geq 0, \\
\forall v \in H^1(0,\pi).
\end{cases}
$$

1.2 A simply supported, or clamped, slender beam subject to elastic obstacles

$$
\begin{cases}
\displaystyle\int_0^a u''(v-u)'' - \lambda \int_0^a \frac{u'}{\sqrt{1+u'^2}}(v-u)' \\
\displaystyle+\Big[\int_{I_1} k_1(v^-)^\gamma dx + \int_{I_2} k_2(v^+)^\beta dx\Big] \\
\displaystyle-\Big[\int_{I_1} k_1(u^-)^\gamma + \int_{I_2} k_2(u^+)^\beta dx\Big] \;\geq\; 0,\; \forall v \in E, \\
u \in E.
\end{cases}
$$

$[0,a]$ $(a>0)$ is the interval occupied by the beam

$E = H_0^2(0,a)$, or $E = H^2(0,a) \cap H_0^1(0,a)$ depending on whether the beam is clamped or is simply supported at the ends 0 and a.

$I_1, I_2 \subset (0,a), |I_1|, |I_2| > 0$ are closed sets representing the domain of possible contact between the beam and the foundations.

1.3 Bifurcation problems for Navier-Stokes flows

$$\begin{cases} \nu \int_\Omega Du : D(v-u) + b(u,u,v-u) + j(v) - j(u) \\ \geq \int_\Omega g(x,u,\lambda) \cdot (v-u),\ \forall v \in E \\ u \in E. \end{cases}$$

$E = \{v \in [H_0^1(\Omega)]^3$: div$v = 0$ a.e. in $\Omega\}$.

$Du = [\partial_i u_j]_{1 \leq i,j \leq 3}$

$\nu > 0$ is the viscosity constant.

b is the trilinear form defined on $[H_0^1(\Omega)]^3$ by

$$\begin{aligned} b(u,v,w) &= \int_\Omega \sum_{i,j=1}^3 u_i(\partial_i v_j) w_j dx \\ &= \int_\Omega u^T (Du) w dx, \end{aligned}$$

for all $u, v, w \in [H_0^1(\Omega)]^3$.

$j : V \to [0, \infty]$ a convex, lower semicontinuous functional such that $j(0) = 0$

$$g : \Omega \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3,\ (x, u, \lambda) \mapsto g(x, u, \lambda)$$

satisfies the Carathéodory condition and g is differentiable with respect to u and $g, D_u g$ satisfies the usual growth condition:

$$\begin{cases} |g(x,u,\lambda)| & \leq A(\lambda) + B(\lambda)|u|^{s-1} \\ |D_u g(x,u,\lambda)| & \leq A(\lambda) + B(\lambda)|u|^{s-2}, \end{cases} \quad (1)$$

for a.e. $x \in \Omega$, all $u, \lambda \in \mathbb{R}$, with $A, B \in L^\infty_{loc}(\mathbb{R})$, $1 < s < 3(2^* = 6)$.

u is the velocity of the fluid

g is the outer force acting on the fluid. g depends on u (in a nonlinear manner) and on λ, which usually represents the magnitude of the force.

$$g(x, 0, \lambda) = 0 \text{ for a.e. } x \in \Omega, \quad \text{all } \lambda \in \mathbb{R},$$

i.e., we have no external force at points with zero velocity

j is a constraint imposed on the velocity. In many cases, j is of the form $j = I_K$, where K is a closed, convex subset of V, representing the set of admissible velocity fields of the fluid. For example, interesting choices of K are the following:

$$K = \{u \in E : u_1(x) \geq -c,\ u_2(x) \geq -d,\ c, d \geq 0\},$$

$$K = \{u \in E : |\nabla \times u| \leq c,\ c \geq 0\},$$

$$K = \{u \in E : |\int_S u \cdot ndS| \leq c,\ c \geq 0\}.$$

In the case $j = 0$, the variational inequality becomes the equation:

$$\begin{cases} \nu \int_\Omega Du : Dv + b(u,u,v) = \int_\Omega g(x,u,\lambda) \cdot v,\ \forall v \in E \\ u \in E, \end{cases}$$

which is the usual variational form of the Navier-Stokes equation (cf. [11], [16], or [17]).

Other interesting choices for the functional j (the case of visco plastic Bingham fluids, cf. [11]) are:

$$j(u) = \int_\Omega \mu(x)|Du|^\gamma,$$

$$j(u) = \int_\Omega \mu(x)|\sum \epsilon_{ij}^2(u)|^\gamma,$$

where

$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

and μ is a nonnegative locally integrable function.

1.4 Bifurcation problems associated with the p-Laplace operator

In this example, we consider bifurcation problems for the following variational inequality:

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u) - \int_\Omega [\lambda |u|^{p-2}u + g(x,u,\lambda)](v-u) \\ +j(v) - j(u) \geq 0, \ \forall v \in E \end{cases}$$

$p > 1$

Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary,

$$E = \{u \in W^{1,p}(\Omega) : v = 0 \ \text{ on } \ \Gamma\},$$

where Γ is a (relatively) open subset of $\partial\Omega$ with positive measure. $W^{1,p}(\Omega)$ is the usual Sobolev space, equipped with the norm,

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega (|u|^p + |\nabla u|^p)\right]^{1/p},$$

$$\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}, \ u \in E,$$

is a norm on E, equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$.

$$g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function, such that

$$g(x, u, \lambda) = o(|u|^{p-1}),$$

as $u \to 0$, uniformly a.e. with respect to $x \in \Omega$ and uniformly with respect to λ on bounded intervals, and, moreover, g satisfies the growth condition

$$|g(x, u, \lambda)| \leq C(\lambda)[m(x) + M|u|^{p-1}],$$

for a.e. $x \in \Omega$, all $u, \lambda \in \mathbb{R}$, where $C(\lambda) \geq 0$ is bounded on bounded sets, $m \in L^{\frac{p}{p-1}}(\Omega)$, and $M > 0$ is a constant.

j is given by

$$j(u) = \int_{\partial\Omega \setminus \Gamma} \psi(u(x)) dS, \; u \in V,$$

where $\psi : \mathbb{R} \to [0, \infty]$ is a proper, convex, lower semicontinuous function.

2 The abstract setting

E a reflexive Banach space

E^* its dual

The norm in E is $\| \cdot \|$

The norm in E^* is $\|\cdot\|_*$.

The pairing between E^* and E is $\langle\cdot,\cdot\rangle$, i.e. if $f \in E^*$ and $u \in E$, then $f(u) = \langle f, u\rangle$.

We shall assume that:

$$j, J : E \longrightarrow \mathbb{R}_+ \cup \infty$$

are convex and lower semicontinuous functionals with

$$j(0) = J(0) = 0.$$

$$A, \alpha : E \longrightarrow E^*$$

are continuous and bounded operators with

$$A(0) = \alpha(0) = 0,$$

which are strictly monotone, coercive and belong to class (S), i.e:

A *is strictly monotone:*

$$\langle A(u) - A(v), u - v\rangle > 0, \text{ whenever } u \neq v.$$

A *is coercive:* There exist constants $c > 0$ and $p > 1$ such that

$$\langle A(u), u\rangle \geq c\|u\|^p, \ \forall u \in E.$$

A *belongs to class* (S) : For all weakly convergent sequences $\{v_n\}$, $v_n \rightharpoonup v$, with

$$\lim\langle A(v_n), v_n - v\rangle = 0,$$

it must hold that

$$v_n \to v.$$

$$B, f : \mathbb{R} \times E \to E^*$$

are completely continuous operators with

$$B(\lambda, 0) = 0 = f(\lambda, 0),\ \forall\lambda \in \mathbb{R}.$$

2.1 Homogenizations

The following relationships between the operators introduced above will be assumed:

For all sequences $\{v_n\},\ v_n \to v$, and all sequences of positive numbers $\sigma_n,\ \sigma_n \to 0+$,

$$\lim \frac{1}{\sigma_n^{p-1}} A(\sigma_n v_n) = \alpha(v).$$

For all weakly convergent sequences $\{v_n\},\ v_n \rightharpoonup v$, and all sequences of positive numbers $\sigma_n,\ \sigma_n \to 0+$, all sequences $\{\lambda_n\},\ \lambda_n \to \lambda$,

$$\lim \frac{1}{\sigma_n^{p-1}} B(\lambda_n, \sigma_n v_n) = f(\lambda, v).$$

For all weakly convergent sequences $\{v_n\},\ v_n \rightharpoonup v$, and all sequences of positive numbers $\sigma_n,\ \sigma_n \to 0+$,

$$\liminf \frac{1}{\sigma_n^p} j(\sigma_n v_n) \geq J(v),$$

further, for all $v \in E$, and all sequences of positive numbers $\sigma_n,\ \sigma_n \to 0+$, there exists a sequence $\{v_n\},\ v_n \to v$, such that

$$\lim \frac{1}{\sigma_n^p} j(\sigma_n v_n) = J(v).$$

2.2 Equivalent operator equations

Consider, for $f \in E^*$, the variational inequality

$$\begin{cases} \langle A(u) - f, v - u\rangle + j(v) - j(u) \geq 0, \\ \forall v \in E. \end{cases}$$

Classical results (see e.g. [11])

$$\Downarrow$$

$$T_{A,j} : E^* \to E$$

by

$$T_{A,j}(f) = u,$$

where u is the unique solution of the inequality.

This operator is also continuous (cf. [9]). Therefore u solves

$$\begin{cases} \langle A(u) - B(\lambda, u), v - u\rangle + j(v) - j(u) \geq 0, \\ \forall v \in E, \end{cases}$$

if and only if u solves

$$T_{A,j}B(\lambda, u) = u.$$

And similarly if we consider the variational inequality

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v - u \rangle + J(v) - J(u) \geq 0, \\ \forall v \in E, \end{cases}$$

then u solves if and only if u solves

$$T_{\alpha,J}f(\lambda, u) = u.$$

It follows from the relationships between A and α, B and f and j and J, that if u solves then so does σu for any $\sigma(> 0) \in \mathbb{R}$.

3 Global bifurcation

Let us assume that $(\lambda_0, 0) \in \mathbb{R} \times E$ is a bifurcation point, then it follows that the homogenized inequality will have a nontrivial solution for $\lambda = \lambda_0$. Therefore, if $a \in \mathbb{R}$ is such that the homogenized inequality has only the trivial solution for $\lambda = a$, it will follow that for $r > 0$, sufficiently small, the Leray-Schauder degree

$$d(\text{id} - T_{\alpha,J}f(a, \cdot), B_r(0), 0)$$

is defined (here $B_r(0)$ is the open ball of radius r in E centered at 0)

and we obtain

$$d(\text{id} - T_{\alpha,J}f(a, \cdot), B_r(0), 0) = d(\text{id} - T_{A,j}B(a, \cdot), B_r(0), 0)$$

(see e.g. [9]).

We hence may employ the homotopy invariance principle of the Leray-Schauder degree, to conclude that if $a, b \in \mathbb{R},\ a < b$ are such that

$$d(\text{id} - T_{\alpha,J} f(a,\cdot), B_r(0), 0) \neq d(\text{id} - T_{\alpha,J} f(b,\cdot), B_r(0), 0)$$

then $[a,b] \times \{0\}$ will contain a bifurcation point. In fact, we may employ the global bifurcation result of Rabinowitz [15] to conclude that global bifurcation takes place in the sense of that theorem.

Thus in such bifurcation problems, in order to be able to apply the above considerations we need to compute the operators α and f, the functional J and verify that the degree changes as λ varies from a to b. This we shall do for the examples considered earlier.

4 Examples revisited

4.1 Semilinear problems

$$a : E \times E \to \mathbb{R}$$

is a continuous, coercive and bilinear form

$$A : E \to E^*$$

is defined by

$$\langle A(u), v\rangle = a(u,v).$$

$$B(\lambda, u) = \lambda Bu + R(u),\ R(u) = o(\|u\|),\ \text{as } u \to 0,$$

with B compact linear

$$j = I_K,$$

where K is a closed convex subset of E with $0 \in K$.

One computes

$p = 2$

$\alpha = A$

$f(\lambda u) = \lambda Bu$

$J = I_{K_0}$, where K_0 is the support cone of K, i.e

$$K_0 = \overline{\cup_{t>0} tK}.$$

If it is the case that K_0 is a subspace of E, then the variational inequality becomes

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v - u \rangle + I_{K_0}(v) - I_{K_0}(u) \geq 0, \\ \forall v \in E, \end{cases}$$

which is equivalent to

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v - u \rangle \geq 0,\ u \in K_0 \\ \forall v \in K_0, \end{cases}$$

and, since K_0 is a subspace, the latter is equivalent to

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v \rangle = 0, \ u \in K_0 \\ \forall v \in K_0. \end{cases}$$

From this we see that the solution operator $T_{\alpha,J}$ is a bounded linear operator and the operator equation becomes

$$u = \lambda T_{\alpha,J} B u.$$

Hence the possible bifurcation points are to be sought among the countable set $\{(\lambda_i, 0)\}$, where λ_i is a characteristic value of the compact linear operator $T_{\alpha,J}B$. And each characteristic value of odd multiplicity will yield a bifurcation point. We note here that what has just been said is true as long as J is the indicator function of a subspace, iregardless whether $j = I_K$ for some closed convex set K.

4.2 A semilinear elliptic problem

$\Omega \subset \mathbb{R}^N$ a bounded domain with smooth boundary $\partial\Omega$, and let $\Gamma \subset \partial\Omega$ be a relatively open subset of positive measure.

$$E = \{u \in H^1(\Omega) : u = 0, \text{ a.e. on } \Gamma\}.$$

Let

$$a : E \times E \to \mathbb{R}$$

be given by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v,$$

a is a continuous, coercive and bilinear form.

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with $g(u) = o(|u|)$ as $u \to 0$, and define $B(\lambda, u)$ by

$$\langle B(\lambda, u), v\rangle = \int_{\Omega} \lambda uv + g(u)v,$$

then

$$\langle f(\lambda, u), v\rangle = \int_{\Omega} \lambda uv.$$

Let the functional j be

$$j(u) = \int_{\partial\Omega} \mu|u|^{\gamma},$$

where μ, γ are positive constants with $1 \leq \gamma < 2$.

Embedding theorems (see [1]) imply that the mapping

$$\begin{array}{ccc} H^1(\Omega) & \hookrightarrow & L^q(\partial\Omega) \\ u & \mapsto & u|_{\partial\Omega} \end{array}$$

are compact for

$$1 \leq q < \bar{p} = \begin{cases} \frac{2(N-1)}{N-2}, & N > 2 \\ \infty, & N = 1, 2 \end{cases}$$

j is convex and lower semicontinuous and (since $p = 2$ and $1 \leq \gamma < 2$) that

$$J(u) = I_{H_0^1(\Omega)}.$$

the inequality is equivalent to the problem

$$\int_\Omega \nabla u \cdot \nabla v + \lambda \int_\Omega uv = 0,\ \forall v \in H_0^1(\Omega),$$

which is equivalent to the eigenvalue problem

$$\Delta u + \lambda u = 0,\ u \in H_0^1(\Omega).$$

Let

$$E = \{u \in W^{1,p}(\Omega) : u = 0,\ \text{a.e. on } \Gamma\}$$

$$A : E \longrightarrow E^*$$

given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v.$$

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with $g(u) = o(|u|^{p-1})$ as $u \longrightarrow 0$, and define $B(\lambda, u)$ by

$$\langle B(\lambda, u), v\rangle = \int_\Omega \lambda |u|^{p-2} uv + g(u)v,$$

$$\langle f(\lambda, u), v\rangle = \int_\Omega \lambda |u|^{p-2} uv.$$

Let j be

$$j(u) = \int_{\partial\Omega} \mu |u|,$$

where μ is a positive constant.

$$\begin{array}{rcl} W^{1,p}(\Omega) & \hookrightarrow & L^1(\partial\Omega) \\ u & \mapsto & u|_{\partial\Omega} \end{array}$$

is compact.

j is convex and lower semicontinuous and that

$$J(u) = I_{W_0^{1,p}(\Omega)}.$$

The inequality is equivalent to the problem

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v + \lambda \int_\Omega |u|^{p-2} uv = 0,\ \forall v \in W_0^{1,p}\Omega,$$

which is equivalent to the eigenvalue problem

$$-\text{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0,\ u \in W_0^{1,p}(\Omega).$$

4.3 Stationary Navier-Stokes flows

A is given by a continuous, coercive and bilinear form, hence $A = \alpha$.

$$\langle f(\lambda, u), v\rangle = \lambda \int_\Omega D_u g(x,0) u \cdot v$$

If $j = I_K$, where K is any of the choices given earlier, then $J = I_E$, since the support cone of K in any of the cases is the whole space.

The homogenized inequality is given by

$$\begin{cases} \nu \int_\Omega Du : Dv + \lambda \int_\Omega D_u g(x,0) u \cdot v = 0 \\ \forall v \in E, \end{cases}$$

which is the eigenvalue problem for the Stokes equation. Its eigenvalues of odd multiplicity hence yield global bifurcation points.

j is given by

$$j(u) = \int_\Omega \mu(x) |Du|^\gamma,$$

where $\mu \in L^\infty(\Omega)$ and $\gamma \geq 1$.

The effective domain of j is given by

$$\{u : j(u) < \infty\} = \begin{cases} E, \ 1 \leq \gamma \leq 2 \\ u \in E : \mu |Du|^\gamma \in L^1(\Omega), \ \gamma > 2 \end{cases}$$

$$J = \begin{cases} I_W, & 1 \leq \gamma < 2 \\ j, & \gamma = 2 \\ I_E, & \gamma > 2, \end{cases}$$

where

$$W = \{u \in E : Du = 0, \ \text{a.e. on } \Omega \setminus \Omega_0\},$$

and

$$\Omega_0 = \{x \in \Omega : \mu(x) = 0\}.$$

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