Young Measures and Nonconvex Variational Problems*

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Abstract

We review the basic facts from Functional Analysis related to the study of oscillation and concentration effects developed by weakly converging sequences in L^p spaces. These effects are displayed by minimizing sequences of certain variational functionals. In order to describe the limiting behavior of such quantities we introduce the notion of Young measure and discuss its basic properties. The last part of these lectures is devoted to some applications to the study of nonconvex variational problems.

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WEAK CONVERGENCE IN L^p

$1<p<\infty$

$$L^p(\Omega;\mathbb{R}^d) := \{u:\Omega\to\mathbb{R}^d \text{ measurable}: \int_\Omega |u(x)|^p dx < +\infty\}$$

$$\|u\|_{L^p} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$$

$p=\infty$

$$L^\infty(\Omega;\mathbb{R}^d) := \{u:\Omega\to\mathbb{R}^d \text{ measurable}: \exists C>0 \text{ s.t } |u(x)|\le C \ \mathcal{L}^N\text{-a.e. } x\in\Omega\}$$

$$\|u\|_{L^\infty} := \inf\{C>0 : |u(x)|\le C \ \mathcal{L}^N\text{-a.e. } x\in\Omega\}$$

$$u_j \rightharpoonup u \text{ in } L^p(\Omega) \quad \boxed{d=1}$$

$(\overset{*}{\rightharpoonup} \text{ if } p=\infty)$

if and only if $\int_\Omega u_j \varphi\, dx \to \int u\varphi\, dx$, $\forall \varphi\in L^{p'}(\Omega)$

$\frac{1}{p}+\frac{1}{p'}=1$, $1'=\infty$, $\infty'=1$

WEAK CONVERGENCE IN L^p

$p > 1$ $\quad u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^d)$

$\left(\overset{*}{\rightharpoonup} \text{ if } p = \infty \right)$

if and only if

$$\begin{cases} \text{(i)} \displaystyle\int_A u_j(x)\,dx \to \int_A u(x)\,dx \\ \quad \text{for all Borel } A \subseteq \Omega \\ \text{(ii)} \displaystyle\sup_j \|u_j\|_{L^p} < \infty \end{cases}$$

$p = 1$ $\quad u_j \rightharpoonup u$ in $L^1(\Omega,\mathbb{R}^d)$

if and only if

$$\int_A u_j(x)\,dx \longrightarrow \int_A u(x)\,dx$$

for all Borel $A \subseteq \Omega$

Vitali's convergence theorem:

$1 \le p < \infty$: $u_j \to u$ in $L^p(\Omega;\mathbb{R}^d)$ if and only if

$$\begin{cases} (1)\ u_j \to u \text{ in measure, and} \\ (2)\ \{u_j\}_{j\geq 1} \text{ is } p\text{-equiintegrable} \end{cases}$$

Note: Important that $\mathcal{L}^N(\Omega) < \infty$

$u_j \to u$ in $L^p \Rightarrow u_j \rightharpoonup u$ in L^p

$\not\Leftarrow$ weak convergence does not imply (1) and (2)

TERMINOLOGY: if $u_j \rightharpoonup u$ in L^p ($\overset{*}{\rightharpoonup}$ if $p=\infty$), then

- $\{u_j\}_{j\geq 1}$ OSCILLATES if (1) fails
- $\{u_j\}_{j\geq 1}$ CONCENTRATES if (2) fails

OSCILLATION

Riemann-Lebesgue's lemma:

Assume $1 \leq p \leq \infty$

- $u \in L^p_{loc}(\mathbb{R}^N, \mathbb{R}^d)$ $[0,1]^N$ periodic
- $u_j(x) := u(jx)$, $x \in \Omega$

Then

$$u_j \rightharpoonup \int_{(0,1)^N} u(x)\,dx \quad \text{in } L^p(\Omega, \mathbb{R}^d)$$

$$(\overset{*}{\rightharpoonup} \text{ if } p = \infty)$$

Remark:

- $\{u_j\}$ oscillates unless $u \equiv$ constant
- $\{u_j\}$ does NOT concentrate

CONCENTRATION

De la Vallée-Poussin :

Assume $1 \le p < \infty$

Then

$\{u_j\}$ p-equiintegrable

$$\Updownarrow$$

there exists $\theta : [0,\infty) \to [0,\infty)$

such that

$$\frac{\theta(t)}{t} \to \infty \text{ as } t \to \infty$$

and

$$\sup_j \int_\Omega \theta(|u_j|^p)\,dx < \infty$$

1. Rademacher's functions on $(0,1)$

$$u_1(x) = \begin{cases} 1 ; & 0 < x < \frac{1}{2} \\ -1 ; & \frac{1}{2} \le x < 1 \end{cases}$$

$$u_j(x) := u_1(jx) \quad , \quad x \in (0,1)$$

Then $u_j \xrightarrow{*} 0$ in $L^\infty(0,1)$

$u_j \not\to 0$ in $L^p(0,1)$ $\forall p$

Oscillation , no concentration

2. $u_j : (0,1) \longrightarrow \mathbb{R}$

$$u_j(x) = \cos(2\pi j x)$$

Then

- $u_j \rightharpoonup 0$ in $L^p(0,1)$
  ($\xrightarrow{*}$ if $p = \infty$)
- $u_j \not\to 0$ in $L^p(0,1)$, $\forall p$

Oscillation, no concentration

3. $\rho : \mathbb{R}^N \longrightarrow \mathbb{R}$ mollifier kernel
i.e. smooth, $\rho \geq 0$, $\operatorname{supp} \rho \subset B(0,1)$
and $\int_{\mathbb{R}^N} \rho \, dx = 1$

$$u_j(x) := j^{\frac{N}{p}} \rho(jx)^{\frac{1}{p}}, \quad x \in \Omega$$

If $1 < p < \infty$ and $0 \in \overline{\Omega}$, then

- $u_j \longrightarrow 0$ in measure
- $u_j \rightharpoonup 0$ in $L^p(\Omega)$
- $u_j \not\longrightarrow 0$ in $L^p(\Omega)$

Concentration, no oscillation

4. $1 < p < \infty$, $u_j : (0,1) \to \mathbb{R}$

$$u_j(x) := \begin{cases} \left(\frac{1}{2} j^2\right)^{\frac{1}{p}} ; & x \in \left(\frac{k}{j+1} - \frac{1}{j^3}, \frac{k}{j+1} + \frac{1}{j^3}\right) \\ & k = 1, 2, \ldots, j \\ 0 & ; \text{ else} \end{cases}$$

Then

- $u_j \to 0$ in measure
- $u_j \rightharpoonup 0$ in $L^p(0,1)$
- $u_j \not\to 0$ in $L^p(0,1)$

Concentration, no oscillation

Riesz representation theorem:

$$C^0(\bar{\Omega})' \cong \mathcal{M}(\bar{\Omega})$$

If $T : C^0(\bar{\Omega}) \longrightarrow \mathbb{R}$ continuous and linear then $\exists!$ signed measure μ on $\bar{\Omega}$, such that

$$T(\varphi) = \langle \mu, \varphi \rangle = \int_{\bar{\Omega}} \varphi \, d\mu^+ - \int_{\bar{\Omega}} \varphi \, d\mu^-$$

for all $\varphi \in C^0(\bar{\Omega})$

Chacon's biting lemma:

Assume $\sup_j \int_\Omega |u_j| dx < \infty$

Then $\exists$ a subsequence $\{u_{j_k}\}$, a map $u \in L^1(\Omega, \mathbb{R}^d)$ and open sets $E_\ell \subset \Omega$ $E_\ell \supseteq E_{\ell+1}$, $\mathcal{L}^N(E_\ell) < \frac{1}{\ell}$, $\ell = 1, 2, \ldots$ such that

$$u_{j_k} \longrightarrow u \text{ in } L^1(\Omega \setminus E_\ell, \mathbb{R}^d)$$

for each fixed ℓ.

REDUCED DEFECT MEASURE

Suppose

- $u_j \longrightarrow u$ in L^p
- $|u_j|^p \mathcal{L}^N \xrightarrow{*} \mu$ in $C^0(\bar{\Omega})'$
- $|u_j|^p \xrightarrow{b} f$

Then

$$\lambda := \mu - f\mathcal{L}^N \geqslant 0$$

is the reduced defect measure

<u>THEOREM</u>:

$\{u_j\}$ p-equiintegrable

if and only if

$$\lambda \equiv 0$$

COROLLARY:

If $u_j \rightharpoonup u$ in $L^p(\Omega; \mathbb{R}^d)$, then there exists a subsequence $\{u_{j_k}\}_{k \geq 1}$, sequences $\{g_k\}$ and $\{b_k\}$, such that

- $u_{j_k} = u + g_k + b_k$,
- $g_k \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^d)$,
- $\{g_k\}$ p-equiintegrable,
- $b_k \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^d)$,
- $b_k \longrightarrow 0$ in measure.

<u>Proof</u>: W.l.o.g. $u \equiv 0$.

$$\sup_j \int_\Omega |u_j|^p dx < +\infty$$

• Chacon: $\exists$ subseq (relabel) $\{u_j\}$, "bits" $E_K \subset \Omega$, $K = 1, 2, \ldots$ and $f \in L^1(\Omega)$ s.t. $|u_j|^p \rightharpoonup f$ in $L^1(\Omega \setminus E_K)$, $\forall K$

• Define $\boxed{g_{j,K} := u_j \chi_{\Omega \setminus E_K}}$. then

$$\lim_{K \to \infty} \lim_{j \to \infty} \int_\Omega \varphi |g_{j,K}|^p dx = \int_\Omega \varphi f \, dx$$

$$\forall \varphi \in L^\infty(\Omega)$$

• $C^0(\bar{\Omega})$ separable. Diagonalization argument: $\exists \{j_K\} \nearrow \infty$ s.t. $\int_\Omega \varphi |g_{j_K,K}|^p dx \to \int_\Omega \varphi f \, dx$ $\forall \varphi \in C^0(\bar{\Omega})$

• Define $\begin{cases} g_K := g_{j_K,K} \\ b_K := u_{j_K} - g_K, \quad K = 1, 2, \ldots \end{cases}$

$1 < p < \infty$

① $u_j(x) = \cos(2\pi j x)$, $x \in (0,1)$

- $|u_j|^p \xrightarrow{b} \int_0^1 |\cos(2\pi x)|^p\,dx$, $E_\ell = \emptyset$
- $|u_j|^p \mathcal{L}^1 \xrightarrow{*} \left(\int_0^1 |\cos(2\pi x)|^p\,dx\right)\mathcal{L}^1$ in $C^0([0,1])'$

$\boxed{\lambda = 0}$

② $0 \in \overline{\Omega}$, ρ a mollifier kernel

$$u_j(x) = j^{\frac{N}{p}} \rho(jx)^{\frac{1}{p}}, \quad x \in \Omega$$

- $|u_j|^p \xrightarrow{b} 0$, $E_\ell = B(0, \frac{1}{\ell})$
- $|u_j|^p \mathcal{L}^N \xrightarrow{*} \delta_0$ in $C^0(\overline{\Omega})'$

$\boxed{\lambda = \delta_0}$

③ $$u_j(x) = \begin{cases} \left(\frac{1}{2} j^2\right)^{\frac{1}{p}}; & x \in \left(\frac{k}{j+1} - \frac{1}{j^3}, \frac{k}{j+1} + \frac{1}{j^3}\right) \\ 0; & \text{else} \end{cases} \quad k = 1, \dots, j$$

- $|u_j|^p \xrightarrow{b} 0$, $E_\ell = \bigcup_{j \geq \ell} \{u_j \neq 0\}$
- $|u_j|^p \mathcal{L}^1 \xrightarrow{*} \mathcal{L}^1$ in $C^0([0,1])'$. $\boxed{\lambda = \mathcal{L}^1}$

Compactifications of $\mathbb{R}^d$

DEF. A (Hausdorff) compactification $\alpha\mathbb{R}^d$ of $\mathbb{R}^d$ is a compact (Hausdorff) space $\alpha\mathbb{R}^d$ and an embedding $\alpha : \mathbb{R}^d \to \alpha\mathbb{R}^d$ such that $\overline{\alpha(\mathbb{R}^d)} = \alpha\mathbb{R}^d$

Examples: 1 One-point compactification (Alexandroff, 1924)

$\mathbb{R}^d \cup \{\infty\}$ with topology

$$O \subseteq \mathbb{R}^d \cup \{\infty\} \text{ open}$$

if either

$$O \subseteq \mathbb{R}^d \text{ and } O \text{ open in } \mathbb{R}^d$$

or

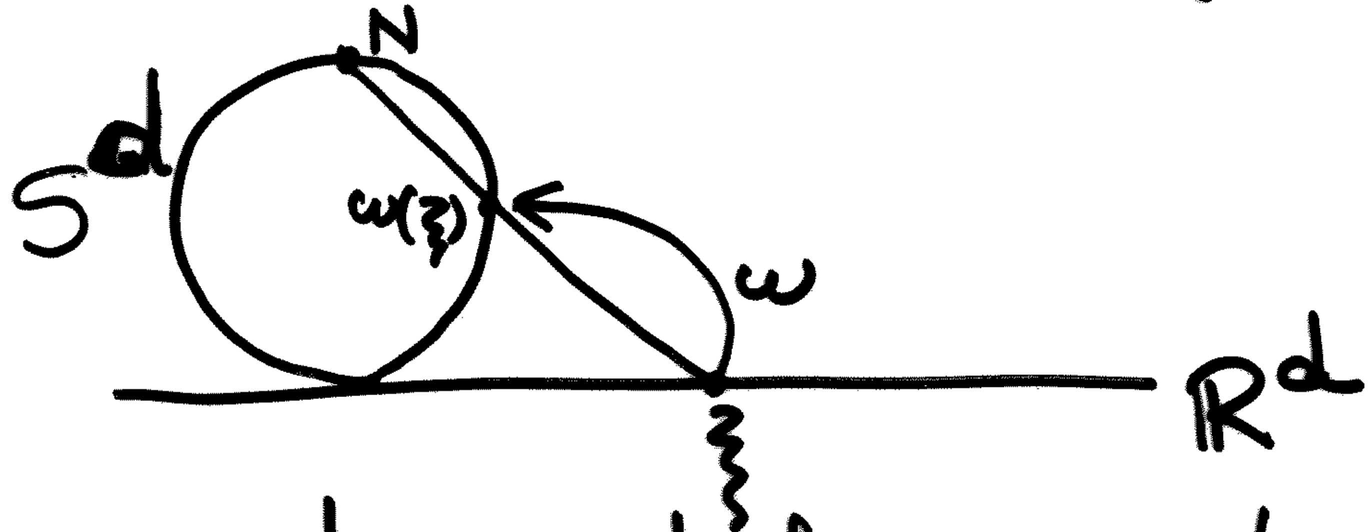
$$\infty \in O \text{ and } \mathbb{R}^d \setminus O \text{ compact in } \mathbb{R}^d$$

$\omega : \mathbb{R}^d \to \mathbb{R}^d \cup \{\infty\}$ the inclusion mapping

$F \in BC(\mathbb{R}^d)$ admits continuous extension to $\mathbb{R}^d \cup \{\infty\}$ if and only if $F(\xi) \to \ell \in \mathbb{R}$ as $|\xi| \to \infty$

Alternatively,

use STEREOGRAPHIC PROJECTION



$\omega : \mathbb{R}^d \longrightarrow S^d$ homeomorphism

onto $S^d \setminus \{N\}$

$\omega(\mathbb{R}^d) = S^d \setminus \{N\}$

$\overline{\omega(\mathbb{R}^d)} = \boxed{S^d =: \omega\mathbb{R}^d}$

2 Sphere compactification

$\gamma : \mathbb{R}^d \longrightarrow B^d$ homeomorphism

$$\gamma(\xi) = \frac{\xi}{1+|\xi|}$$

$$\overline{\gamma(\mathbb{R}^d)} = \overline{B^d} =: \gamma\mathbb{R}^d$$

$F \in BC(\mathbb{R}^d)$ extends by continuity to $\overline{B^d}$ if and only if $\lim\limits_{\substack{t\to\infty \\ \xi'\to\xi}} F(t\xi')$ exists and defines a continuous function of $\xi \in \mathbb{R}^d \setminus \{0\}$

$\{F \in C^0(\mathbb{R}^d) : \exists$ 0-homogeneous, continuous $G : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ s.t. $F(\xi) - G(\xi) \longrightarrow 0$ as $|\xi| \to \infty\} =$

$$= \{ F|_{\mathbb{R}^d} : F \in C^0(\overline{B^d}) \}$$

Tychonoff's construction :

- $\mathcal{F} \subseteq BC(\mathbb{R}^d)$

Define

$$e_{\mathcal{F}} : \mathbb{R}^d \longrightarrow \omega\mathbb{R}^d \times \prod_{f \in \mathcal{F}} \overline{\operatorname{im} f}$$

by

$$\begin{cases} e_{\mathcal{F}}(\xi)(\omega) := \omega(\xi) \\ e_{\mathcal{F}}(\xi)(f) := f(\xi) \, , \quad f \in \mathcal{F} \end{cases}$$

Then $e_{\mathcal{F}}$ is an embedding,

$e_{\mathcal{F}}\mathbb{R}^d := \overline{e_{\mathcal{F}}(\mathbb{R}^d)}$ compact and Hausdorff

R.E. Chandler
"Hausdorff compactifications"
L.N.P.A.M, vol 23
Marcel Dekker, Inc., 1976

Properties

! (1) Each $f \in \mathcal{F}$ has a cont. extension $\bar{f} : e_{\mathcal{F}}\mathbb{R}^d \longrightarrow \mathbb{R}$:

$$\pi_f\big|_{e_{\mathcal{F}}\mathbb{R}^d} : e_{\mathcal{F}}\mathbb{R}^d \longrightarrow \mathbb{R}$$

continuous,

$$\pi_f \circ e_{\mathcal{F}} = f$$

(2) $e_{\mathcal{F}}\mathbb{R}^d$ is minimal with respect to (1)

$\omega\mathbb{R}^d$ embeds into $e_{\mathcal{F}}\mathbb{R}^d$

! (3) The remainder $S = e_{\mathcal{F}}\mathbb{R}^d \setminus \mathbb{R}^d$ is compact

(4) All Hausdorff compactifications of $\mathbb{R}^d$ can be obtained using Tychonoff's construction

EXAMPLES:

1) $\mathcal{F} = \emptyset \Rightarrow$ one-point compactification

$\mathbb{R}^d \cup \{\infty\} = \omega\mathbb{R}^d$ compact, metrizable

2) $\mathcal{F} = \{ f \in BC(\mathbb{R}^d) : f(\xi) - g(\xi) \to 0$ as $|\xi| \to \infty$, $g : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ cont., 0-homogeneous $\}$ separable

$\Rightarrow$ sphere compactification

$\gamma\mathbb{R}^d = \overline{B}^d$ compact, metrizable

3) $\mathcal{F} = BC(\mathbb{R}^d) \Rightarrow$ Stone-Čech compactification

$\beta\mathbb{R}^d := e_{BC(\mathbb{R}^d)}\mathbb{R}^d$ compact NOT metrizable

YOUNG measures

- $\mathcal{C}$ class of continuous functions $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ with

$$\left\{ \frac{F}{1+|\cdot|^p} : F \in \mathcal{C} \right\} \subseteq BC(\mathbb{R}^d)$$

<u>separable</u>

- $\overline{\mathbb{R}^d}$ metrizable compactification

$$S = \overline{\mathbb{R}^d} \setminus \mathbb{R}^d \text{ remainder}$$

(compact)

- $\tilde{F}$ = continuous extension of $\frac{F}{1+|\cdot|^p}$ to $\overline{\mathbb{R}^d}$

- $F^\infty := \tilde{F}|_S$, i.e.

$$F^\infty(\xi) = \lim_{\substack{\xi' \to \xi \\ \xi' \in \mathbb{R}^d}} \frac{F(\xi')}{|\xi'|^p} , \xi \in S$$

'recession fctn.'

EXISTENCE THEOREM.

If $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^d)$ then $\exists$ subsequence $\{u_{j_k}\}_k$, with reduced defect measure λ, Borel maps

$$\Omega \ni x \longmapsto \nu_x \in \mathcal{M}_1(\mathbb{R}^d)$$

and

$$\bar{\Omega} \ni x \longmapsto \nu_x^\infty \in \mathcal{M}_1(S),$$

such that

$$F(u_{j_k})\mathcal{L}^N \xrightarrow{*} \left(\int_{\mathbb{R}^d} F(\xi)\,d\nu_x(\xi)\right)\mathcal{L}^N + \left(\int_S F^\infty(\xi)\,d\nu_x^\infty(\xi)\right)\lambda$$

in $C^0(\bar{\Omega})'$, for every $F \in \mathcal{C}$

Terminology: - ν_x Young measure for oscillation

- ν_x^∞, λ Young measure for concentration

Particular cases:

1. $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$

$$F(u_{j_k})\mathcal{L}^N \xrightarrow{*} \int_{\mathbb{R}^d} F\, d\nu_x \mathcal{L}^N + \left(\lim_{|\xi|\to\infty} \frac{F(\xi)}{|\xi|^p}\right)\lambda$$

in $C^0(\overline{\Omega})'$

2. $\overline{\mathbb{R}^d} = \overline{B^d}$

$$F(u_{j_k})\mathcal{L}^N \xrightarrow{*} \int_{\mathbb{R}^d} F\, d\nu_x \mathcal{L}^N + \int_{S^{d-1}} F^\infty d\nu_x^\infty \lambda$$

in $C^0(\overline{\Omega})'$, where

$F^\infty(\xi) = \lim_{t\to\infty} \frac{F(t\xi)}{t^p}$, is p-homogeneous ($F^\infty(t\xi) = t^p F^\infty(\xi)$, $t>0$)

Rmk. In all cases:

- $\int_\Omega \int_{\mathbb{R}^d} |\xi|^p d\nu_x(\xi) \leq \varliminf_{k\to\infty} \int_\Omega |u_{j_k}|^p dx < \infty$

- $\overline{\nu_x} := \int_{\mathbb{R}^d} \xi\, d\nu_x(\xi) = u(x)$ a.e.

Examples (sphere compactification)

① $$\boxed{\begin{array}{l} \bullet\ u \in L^p_{loc}(\mathbb{R}^N, \mathbb{R}^d) \ \ [0,1]^N \text{ periodic} \\ \bullet\ u_j(x) := u(jx), \quad x \in \Omega \end{array}}$$

$$\lambda = 0, \quad \nu_x = \nu, \quad \langle \nu, \phi \rangle = \int_{(0,1)^N} \phi(u(x))\, dx$$

(homogeneous Y.m.)

ν_x^∞ unimportant

② $$\boxed{u_j(x) := -j^{\frac{1}{p}} \chi_{(-\frac{1}{j}, 0)}(x) + j^{\frac{1}{p}} \chi_{(0, \frac{1}{j})}(x), \quad x \in (-1,1)}$$

$$\lambda = 2\delta_0, \quad \nu_x = \delta_0, \quad \nu_x^\infty = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

③ $$u_j(x) := \sum_{k=0}^{j-1} j^{\frac{1}{p}} \chi_{(\frac{k}{j}, \frac{k}{j} + \frac{1}{j^2})}(x) \begin{pmatrix} \cos 2\pi j^2 x, \\ \sin 2\pi j^2 x \end{pmatrix}$$

$u_j : (0,1) \to \mathbb{R}^2$

$$\lambda = \mathcal{L}^1, \quad \nu_x = \delta_0, \quad \nu_x^\infty = \frac{1}{2\pi} \mathcal{H}^1 \llcorner S^1$$

Sketch of proof (follows Alibert & Bouchitté J. Conv. Anal. '97)

I • Define

$$\langle \mathcal{E}_{u_j}, \phi \rangle := \int_\Omega \phi(x, u_j)(1+|u_j|^p)\,dx,$$

$$\phi \in C^0(\bar\Omega \times \overline{\mathbb{R}^d})$$

– $\exists$ subsequence $\{\mathcal{E}_{u_{j_k}}\}_k$ and $\mu \in \mathcal{M}(\bar\Omega \times \overline{\mathbb{R}^d})$

s.t. $\begin{cases} \bullet\ \mathcal{E}_{u_{j_k}} \xrightarrow{*} \mu \text{ in } C^0(\bar\Omega \times \overline{\mathbb{R}^d})' \\ \bullet\ \mu \geq 0 \\ \bullet\ (1+|u_{j_k}|^p)\mathcal{L}^N \xrightarrow{*} \mathrm{proj}_{\#}\mu \\ \qquad \text{in } C^0(\bar\Omega)' \end{cases}$

– let $\tilde\mu := \mathrm{proj}_{\#}\mu$

II $\exists$ Borel map $\bar\Omega \ni x \mapsto \tilde\mu_x \in \mathcal{M}_1(\overline{\mathbb{R}^d})$

s.t.

$$\langle \mu, \phi \rangle = \int_{\bar\Omega} \int_{\overline{\mathbb{R}^d}} \phi(x, \xi)\,d\tilde\mu_x(\xi)\,d\tilde\mu(x)$$

$$\forall\, \phi \in C^0(\bar\Omega \times \overline{\mathbb{R}^d})$$

Disintegration lemma

Let X, Y - compact metric spaces, μ positive finite Borel measure on $X \times Y$,

$\tilde{\mu} = \mathrm{proj}_{X\#}(\mu)$, $\mathrm{proj}_X : X \times Y \to X$

Then

there exists a Borel map

$$X \ni x \longmapsto \mu_x \in \mathcal{M}_1(Y),$$

such that

$$\mu = \int_X \delta_x \otimes \mu_x \, d\tilde{\mu}$$

i.e.

$$\langle \mu, \phi \rangle = \int_X \int_Y \phi(x,y) \, d\mu_x(y) \, d\tilde{\mu}(x)$$

for all $\phi \in C^0(X \times Y)$

(all bounded Borel ϕ)

Furthermore, μ_x is $\tilde{\mu}$-essentially unique.

III • Lebesgue-Radon-Nikodym :

$$\tilde{\mu} = \tilde{a}\,\mathcal{L}^N + \tilde{\mu}^s, \quad 0 \le \tilde{a} \in L^1(\Omega), \quad \tilde{\mu}^s \perp \mathcal{L}^N$$

• <u>claim</u> : (i) $\displaystyle \tilde{a}(x) = \frac{1}{\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^d}\, d\tilde{\mu}_x(\xi)} \geqslant 1$

(ii) $\tilde{\mu}_x(S) = 1$, $\tilde{\mu}^s$ - a.e. $x \in \bar{\Omega}$

• <u>Final claim</u> : there exist

- positive finite measure λ on $\bar{\Omega}$
- Borel maps $\begin{cases} \Omega \ni x \mapsto \nu_x \in \mathcal{M}_1(\mathbb{R}^d) \\ \bar{\Omega} \ni x \mapsto \nu_x^\infty \in \mathcal{M}_1(S) \end{cases}$

such that

$$\langle \mu, \phi \rangle = \int_\Omega \int_{\mathbb{R}^d} \phi(x,\xi)(1+|\xi|^p)\, d\nu_x(\xi)\, dx$$

$$+ \int_{\bar{\Omega}} \int_S \phi(x,\xi)\, d\nu_x^\infty(\xi)\, d\lambda(x), \quad \forall \phi \in C^0(\bar{\Omega} \times \bar{\mathbb{R}}^d)$$

$$\boxed{\lambda := \tilde{\mu}_x(S)\,\tilde{\mu}}$$

$$\langle \nu_x, \phi \rangle := \tilde{a}(x) \int_{\mathbb{R}^d} \frac{\phi(\xi)}{1+|\xi|^p}\, d\tilde{\mu}_x(\xi)$$

$$\forall \phi \in BC(\mathbb{R}^d), \quad \mathcal{L}^N\text{-a.e. } x$$

$$\langle \nu_x^\infty, \phi \rangle := \frac{1}{\tilde{\mu}_x(S)} \int_S \phi \, d\tilde{\mu}_x,$$

$$\forall \phi \in C^0(S) \quad \lambda\text{-a.e. } x \in \bar{\Omega}$$

Sufficiency Theorem

Let

- $\lambda \in C^0(\bar{\Omega})'$, $\lambda \geq 0$
- $\Omega \ni x \longmapsto \nu_x \in \mathcal{M}_1(\mathbb{R}^d)$
- $\bar{\Omega} \ni x \longmapsto \nu_x^\infty \in \mathcal{M}_1(S)$

$\Big\}$ Borel maps

If $1 < p < \infty$ and

$$\int_\Omega \int_{\mathbb{R}^d} |\xi|^p \, d\nu_x(\xi) \, dx < \infty$$

Then

$$\exists \; u_j \xrightarrow[\text{in } L^p(\Omega, \mathbb{R}^d)]{} \bar{\nu}_x := \int_{\mathbb{R}^d} \xi \, d\nu_x(\xi)$$

and

$$\int_\Omega \phi(x, u_j)(1 + |u_j|^p) \, dx \longrightarrow$$

$$\longrightarrow \int_\Omega \int_{\mathbb{R}^d} \phi(x,\xi)(1+|\xi|^p) \, d\nu_x(\xi) \, dx + \int_{\bar{\Omega}} \int_S \phi(x,\xi) \, d\nu_x^\infty(\xi) \, d\lambda(x)$$

$$\forall \; \phi \in C^0(\bar{\Omega} \times \overline{\mathbb{R}^d})$$

Basic properties:

① <u>L.S.C.</u> If $F:\Omega\times\mathbb{R}^d\to[0,\infty]$ is a normal integrand then

$$\varliminf_{j\to\infty}\int_\Omega F(x,u_j)\,dx \geq \int_\Omega\int_{\mathbb{R}^d} F(x,\xi)\,d\nu_x\,dx$$

② <u>Support</u>: If $u_j(x)\in K\subseteq\mathbb{R}^d$ a.e., then $\operatorname{spt}(\nu_x)\subseteq\overline{K}$ a.e.

③ continuity If $F:\Omega\times\mathbb{R}^d\to\mathbb{R}$ Carathéodory and $\sup_j\int_\Omega|F(x,u_j)|\,dx<\infty$ then

$$F(x,u_j)\xrightarrow{b}\int_{\mathbb{R}^d}F(x,\cdot)\,d\nu_x$$

④ $u_j\to u$ in measure if and only if $\nu_x=\delta_{u(x)}$ a.e.

5 $F \in \mathcal{C}$. Then

$$\{F(u_j)\} \text{ equiintegrable } \Longleftrightarrow \langle \nu_x^\infty, |F^\infty| \rangle = 0 \quad \lambda\text{-a.e. } x \in \overline{\Omega}$$

CONVEXITY CONDITIONS

$$f : \mathbb{M}^{d\times N} \longrightarrow \mathbb{R}$$

- CONVEX if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$
  $\forall \lambda \in [0,1],\ A, B \in \mathbb{M}^{d\times N}$

- QUASICONVEX (Morrey 1952)

$$\boxed{\begin{array}{l} f(A) \leq \dfrac{1}{\mathcal{L}^N(D)} \displaystyle\int_D f(A + \nabla\varphi(x))\,dx \\ \forall D \subseteq \mathbb{R}^N,\ \forall A \in \mathbb{M}^{d\times N}, \\ \qquad \forall \varphi \in W_0^{1,\infty}(D, \mathbb{R}^d) \end{array}}$$

- the definition is independent of the domain of integration
- difficult to verify in practice

→ NOT a LOCAL condition

conjectured by Morrey in 1952
confirmed by Jan Kristensen in 1999

LOWER SEMICONTINUITY

(Morrey 1952, Acerbi & Fusco 1984)

$f : \Omega \times \mathbb{R}^d \times \mathbb{M}^{d \times N} \longrightarrow \mathbb{R}$ Carathéodory

$$0 \leq f(x, u, A) \leq C(1 + |A|^p)$$

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx$$

Then I is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^d)$

if and only if

$f(x, u, \cdot)$ is quasiconvex $\forall u \in \mathbb{R}^d$, $\mathcal{L}^N$-a.e. $x \in \Omega$

The Localization Principle

(Kinderlehrer & Pedregal 1991, 1994)

$u_j \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$

$\{\nu_x\}_{x\in\Omega}$ ... the Young measure (for oscillation) generated by $\{\nabla u_j\}$

$$\Downarrow$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$, $\exists\, \varphi_j \in W^{1,p}(\Omega, \mathbb{R}^d)$ $j = 1, 2, \dots$

s.t.

(i) $\varphi_j(y) = \left[\int_{M^{d\times N}} A \, d\nu_x(A)\right] y$, $y \in \partial\Omega$

(ii) the Young measure generated by $\{\nabla\varphi_j\}$ is ν_x (homogeneous)

The Decomposition Lemma

(following Fonseca, Müller & Pedregal
SIAM J. Math. Analysis 1998)

$\{\varphi_j\}$ bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$

There exists $\{\varphi_{j_k}\} \subset \{\varphi_j\}$, and a sequence $\{z_k\} \subset W^{1,p}(\Omega, \mathbb{R}^d)$ such that

(1) $$\mathcal{L}^N\left(\left\{x \in \Omega \,\middle|\, \begin{array}{l} z_k(x) \neq \varphi_{j_k}(x) \text{ or} \\ \nabla z_k(x) \neq \nabla \varphi_{j_k}(x) \end{array}\right\}\right) \to 0$$

as $k \to \infty$

(2) $\{\nabla z_k\}$ is p-equiintegrable

<u>Remark</u>: If Ω has Lipschitz bdry then each z_k may be chosen to be a Lipschitz function!

$f : \mathbb{M}^{d\times N} \longrightarrow \mathbb{R}$ is RANK-ONE CONVEX

if $\boxed{\begin{array}{l} t \longmapsto f(A+tB) \text{ is CONVEX} \\ \forall A, B \in \mathbb{M}^{d\times N},\ \mathrm{Rk}(B) \leq 1 \end{array}}$

$f : \mathbb{M}^{d\times N} \longrightarrow \mathbb{R}$ is POLYCONVEX
(J. BALL 1977)

if $\boxed{\begin{array}{l} f(A) = \text{convex function of} \\ \text{minors of } A \end{array}}$

E.g. if $d = N = 2$, $f(A) = g(A, \det A)$
$d = N = 3$, $f(A) = h(A, \mathrm{cof} A, \det A)$
g, h convex

<u>Rmk</u> g, h above are NOT unique

REMARKS

1) If $d=1$ or $N=1$, then

$$(C) \Longleftrightarrow (P) \Longleftrightarrow (Q) \Longleftrightarrow (R)$$

2) If f is smooth (C^2) then (R) $\Longleftrightarrow$ Legendre–Hadamard:

$$\sum_{i,j=1}^{d} \sum_{\alpha,\beta=1}^{N} \frac{\partial^2 f(X)}{\partial X_\alpha^i \partial X_\beta^j} a^i b_\alpha a^j b_\beta \geq 0$$

$\forall\, a \in \mathbb{R}^d,\ b \in \mathbb{R}^N$

$\forall\, X = \left(X_\alpha^i\right)_{\substack{1 \le i \le d \\ 1 \le \alpha \le N}} \in \mathbb{M}^{d \times N}$

3) (R) $\Longrightarrow$ f is locally Lipschitz

$d, N \geq 2$ The Vectorial Case

Serre, Terpstra ?

$$(C) \underset{\not\Longleftarrow}{\Longrightarrow} (P) \underset{\not\Longleftarrow}{\Longrightarrow} (Q) \underset{\Longleftarrow ?}{\Longrightarrow} (R)$$

"$\Longleftrightarrow$" for quadratic forms on $\mathbb{M}^{2\times 2}$

fails in general even on $\mathbb{M}^{2\times 2}$ (Alibert & Dacorogna, Šverák)

"$\Longleftrightarrow$" true for quadratic forms (see Tartar's 1978 Heriot-Watt notes)

Morrey's Conjecture (1952): $(R) \not\Longrightarrow (Q)$

Quasiconvexity vs. Rank-one convexity

<u>Thm</u> (Šverák, Proc. Royal Soc. Edinburgh 1992)

$$\boxed{\text{If } d \geq 3,\ N \geq 2, \text{ then (R) does not imply (Q)}}$$

For $N \geq d = 2$ NOT KNOWN in general

Recent progress S. Müller, 1999
(R) $\Longrightarrow$ (Q) on 2×2 diagonal matrices

... Later generalized to certain hypersurfaces in $M^{2\times 2}$ by Chaudhury & Müller

Example (Dacorogna & Marcellini)

$d = N = 2$

$$f_\gamma(A) = |A|^4 - 2\gamma |A|^2 \det A$$

- f_γ convex $\iff |\gamma| \le \frac{2\sqrt{2}}{3}$
- f_γ polyconvex $\iff |\gamma| \le 1$
- f_γ quasiconvex $\iff |\gamma| \le 1 + \varepsilon$ for some $\varepsilon > 0$ (unknown)
- f_γ rank-one convex $\iff |\gamma| \le \frac{2}{\sqrt{3}}$

$\frac{2}{\sqrt{3}} \cong 1.1547$

Numerically, $1 + \varepsilon = 1.1547\ldots$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}^{2\times 2}$$

$$A^+ := \frac{1}{2}\begin{pmatrix} a+d & b-c \\ c-d & a+d \end{pmatrix}$$

<u>**conformal**</u> : $(A^+)^T A^+ = \det(A^+)\, I_2$

$$A^- := \frac{1}{2}\begin{pmatrix} a-d & b+c \\ b+c & d-a \end{pmatrix}$$

<u>**anticonformal**</u>: $(A^-)^T A^- = -\det(A^-)\, I_2$

$$A = A^+ + A^-$$

The Burkholder - Šverák function

$$F : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R},$$

$$F(A) := \begin{cases} \det A & \text{if } A \in \mathcal{E} \\ \sqrt{2}\,|A^+| - 1 & \text{if } A \notin \mathcal{E} \end{cases}$$

where

$$\mathcal{E} = \left\{ A \in \mathbb{M}^{2\times 2} \,\middle|\, |A^+| + |A^-| \le \sqrt{2} \right\}$$

Fact 1: F is rank-one convex
Fact 2: F is NOT polyconvex

<u>Open question</u>: Is F quasiconvex?

If NO $\Longrightarrow$ MORREY's conjecture is true in full generality

If YES $\Longrightarrow$ interesting implications in the theory of quasiconformal mappings

In particular, will validate IWANIEC's conjecture on the norm of the AHLFORS – BEURLING transform

<u>Consequence</u>: a stronger form of ASTALA's area distortion thm (Gehring & Reich conjecture, 1966)

(R) $\overset{?}{\Longrightarrow}$ (Q) also important in

- the study of fine differentiability properties of Lipschitz maps (PREISS J. Funct. Analysis 1990)
- the theory of composites