Geometry

Contents

1. Notation

$$G = \mathrm{SL}_2(\mathbb{R})$$

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \right\}$$

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}$$

$$P = \left\{ \begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \right\} = AN$$

$$\overline{P} = \left\{ \begin{bmatrix} a & 0 \\ x & 1/a \end{bmatrix} \right\} = A\overline{N}$$

$$K = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\}$$

Conjugation by w takes an element of P to

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 0 \\ -x & a \end{bmatrix}$$

In particular it acts as involution $a \mapsto a^{-1}$ on A and takes P to $\overline{P}$.

The group N is normal in P and

$$\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a^2 x \\ 0 & 1 \end{bmatrix}$$

If X is a 2×2 matrix then the series

$$\exp X = I + X + \frac{X^2}{2} + \cdots$$

converges. For small ε

$$\exp \varepsilon X = I + \varepsilon X + O(\varepsilon^2)$$

Lemma. *For any X*

$$\det \exp(X) = \exp \mathrm{trace} X$$

The tangent space $\mathfrak{g}$ at I on G may be identified with matrices of trace 0.

$$\exp t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$\exp t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\exp t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\nu_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

2. Complex geometry

The complex projective line is

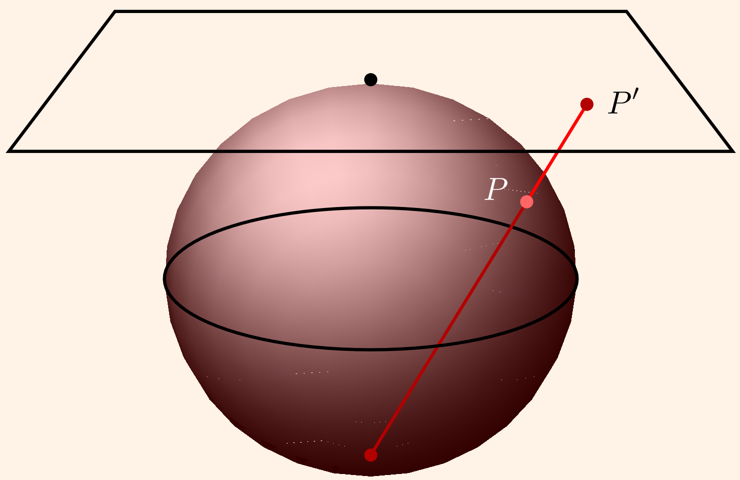
$$\mathbb{P}_{\mathbb{C}} = \mathbb{P}^1(\mathbb{C}) = \mathbb{C}^2 - \{0\}/\mathbb{C}^{\times} \colon (x, y) \longmapsto (\!(x, y)\!)$$

It is covered by two copies of $\mathbb{C}$

$$z \longmapsto (\!(z, 1)\!), \quad (\!(1, z)\!)$$

whose complements are single points $(\!(1, 0)\!)$ **and** $(\!(0, 1)\!)$**.**

$$\mathbb{P}_{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$$



The group G acts on $\mathbb{C}$ by fractional linear transformations:

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} &= \begin{bmatrix} az+b \\ cz+d \end{bmatrix} \\
&= (cz+d) \begin{bmatrix} (az+b)/(cz+d) \\ 1 \end{bmatrix} \\
g \begin{bmatrix} z \\ 1 \end{bmatrix} &= J(g,z) \begin{bmatrix} g(z) \\ 1 \end{bmatrix}
\end{aligned}$$

The function J is called the automorphy factor.

The map $z \mapsto (az+b)/cz+d)$ from $\mathbb{C} \cup \{\infty\}$ to itself is also called a Möbius transformation.

$$
\begin{aligned}
g\begin{bmatrix} z \\ 1 \end{bmatrix} &= J(g,z)\begin{bmatrix} g(z) \\ 1 \end{bmatrix} \\
gh\begin{bmatrix} z \\ 1 \end{bmatrix} &= g\left(J(h,z)\begin{bmatrix} h(z) \\ 1 \end{bmatrix}\right) \\
&= J(g,h(z))J(h,z)\begin{bmatrix} gh(z) \\ 1 \end{bmatrix} \\
&= J(gh,z)\begin{bmatrix} h(z) \\ 1 \end{bmatrix} \\
J(gh,z) &= J(g,h(z))J(h,z)
\end{aligned}
$$

The function $g \mapsto J(g,z)$ is a character of the isotropy *Fix*$($*z*$)$.

The group P is the stabilizer of $(\!(1,0)\!) = \infty$:

$$\begin{bmatrix} a & x \\ 0 & 1/a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The copies of $\mathbb{C}$ are orbits of $\overline{N}_{\mathbb{C}}$ and $N_{\mathbb{C}}$:

$$\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ z \end{bmatrix}$$

This gives us the Bruhat decomposition:

$$\begin{aligned} \mathbb{P}_{\mathbb{C}} &= N_{\mathbb{C}} w(\infty) \cup \{\infty\} \\ G &= NwP \cup P \\ &= PwN \cup PwNw^{-1} \\ &= P\overline{N} \cup PwN \quad (\textbf{open sets}) \end{aligned}$$

Möbius transformations take circles and lines to circles and lines.

$$\begin{aligned} 0 &= \alpha x + \beta y + C \\ &= \operatorname{RE}(\alpha - i\beta)(x + iy)) + C \end{aligned}$$

$$\begin{aligned} 0 &= |z - w|^2 - r^2 \\ &= (z - w)(\overline{z} - \overline{w}) - r^2 \\ &= |z|^2 - 2\operatorname{RE}(z\overline{w}) + |w|^2 - r^2 \end{aligned}$$

$$0 = A|z|^2 + 2\operatorname{RE}(Bz) + C$$

$$0 = \begin{bmatrix} \overline{z} & 1 \end{bmatrix} \begin{bmatrix} A & \overline{B} \\ B & C \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}$$

Line: $A = 0$**, circle:** $A \neq 0$**.**

Circles and lines are the null cones of Hermitian forms H with negative determinants. The stabilizer of the inside of a circle or of a side of a line is a special unitary group $\mathrm{SU}(H)$. The group $\mathrm{SL}_2(\mathbb{R})$ is the special unitary group of

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

and hence stabilizes the upper half plane

$$\mathcal{H} = \{z = x + iy \,|\, y > 0\}\,.$$

($^t\overline{X}CX = C$ **if and only if** $CX = {}^t\overline{X}^{\,-1}C$)

3. The upper half plane

Theorem.

$$y(g(z)) = \frac{y(z)}{|cz+d|^2} = \frac{y(z)}{|J(g,z)|^2}$$

$$\begin{aligned}
y(g(z)) &= \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}\right) \\
&= \frac{1}{2i}\,\frac{(az+b)(c\overline{z}+d)-(a\overline{z}+b)(cz+d)}{|cz+d|^2} \\
&= \frac{(ad-bc)y}{|cz+d|^2}
\end{aligned}$$

So we see again that $\mathrm{SL}_2(\mathbb{R})$ takes $\mathcal{H}$ to itself.

The group K is the isotropy subgroup of i.

$$\frac{ai+b}{ci+d} = i, \quad ai+b = -c+di$$

$$a = d \quad b = -c$$

and the matrix $= \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$

So $\mathcal{H} = G/K$. Since

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix} : i \longmapsto \frac{ai}{1/a} = a^2 i \longmapsto \frac{a^2 i + x}{1} = a^2 i + x$$

the group P acts transitively on $\mathcal{H}$ and $G = PK$.

Iwasawa decomposition: $G = PK$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & (ac+bd)/r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \gamma & -\sigma \\ \sigma & \gamma \end{bmatrix}$$

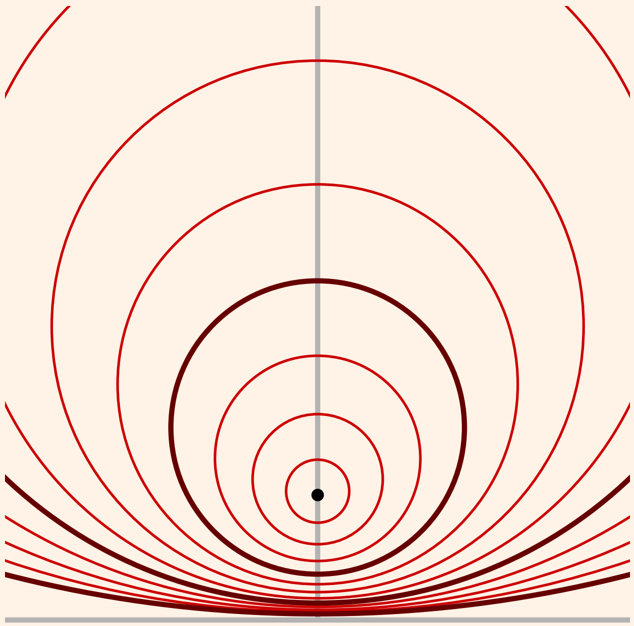
where $r = \sqrt{c^2 + d^2}, \quad \gamma = d/r, \sigma = c/r$.

This is because

$$\begin{aligned} g(i) &= \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} \\ &= \frac{(ac+bd)+i(ad-bc)}{c^2+d^2} = \frac{i+(ac+bd)}{c^2+d^2} \\ &= \alpha^2 i + \chi = p(i) \end{aligned}$$

and solve $g = pk$ to get $k = p^{-1}g$.

The group K fixes i, and its orbits are circles:



The rotation matrix with angle θ rotates by 2θ in the clockwise direction.

Since

$$\frac{dg(z)}{dz} = \frac{1}{(cz+d)^2}, \quad y(g(z)) = \frac{y(z)}{|cz+d|^2}$$

the metric

$$\frac{|dz|^2}{y^2} = \frac{dz \cdot d\overline{z}}{y^2} = \frac{dx^2 + dy^2}{y^2}$$

is G-invariant, as is the differential 2-form

$$\frac{dz \wedge d\overline{z}}{(-2i)\, y^2} = \frac{(dx + i\, dy) \wedge (dx - i\, dy)}{(-2i)\, y^2} = \frac{dx \wedge dy}{y^2}$$

which hence determines a G-invariant measure on $\mathcal{H}$. The Laplacian in this metric is

$$y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

4. The Cayley transform

The Cayley transform

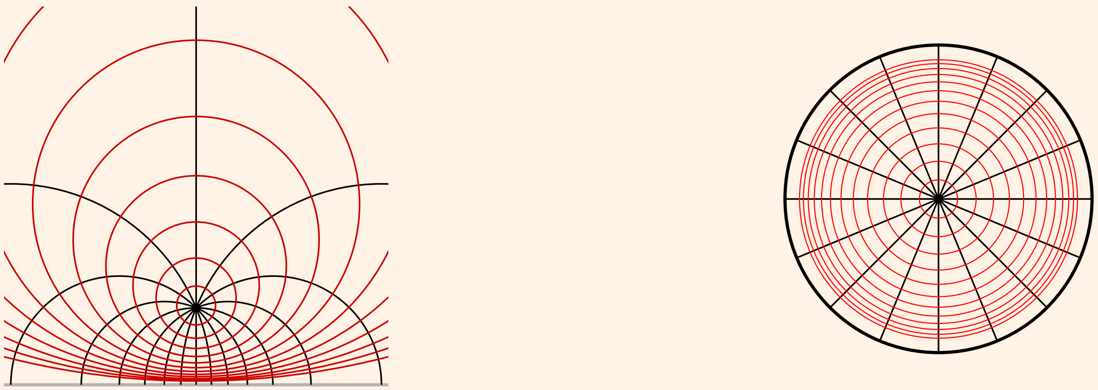
$$z \mapsto \frac{z-i}{z+i}$$

takes $\mathcal{H}$ to

$$\mathbb{D} = \left\{ z \,\middle|\, |z| < 1 \right\}$$

It is the Möbius transformation associated to the matrix

$$C = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$



Any element X of $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{D}$ by conjugation:

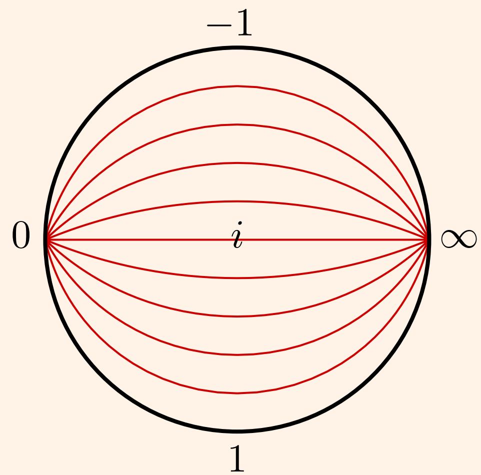
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{1}{2i} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$$

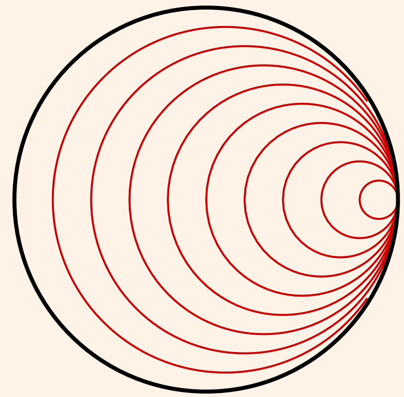
$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \longmapsto \begin{bmatrix} c - is & 0 \\ 0 & c + is \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \longmapsto \begin{bmatrix} \dfrac{a + a^{-1}}{2} & \dfrac{a - a^{-1}}{2} \\ \dfrac{a - a^{-1}}{2} & \dfrac{a + a^{-1}}{2} \end{bmatrix}$$

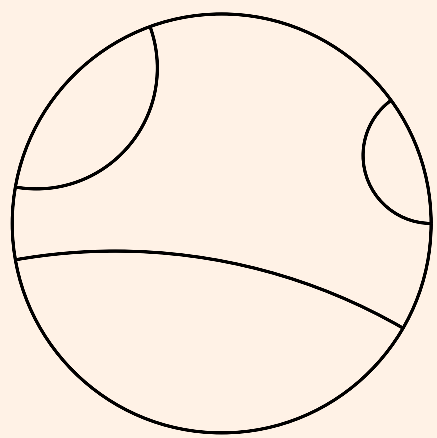
$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 - w & w \\ -w & 1 + w \end{bmatrix} \quad (w = x/2i)$$

Orbits of A and orbits of N:





The group $\mathrm{SL}_2(\mathbb{R})$ acts as non-Euclidean isometries in the Poincaré model, in which geodesics are arcs intersecting the boundary at right angles.



From the action on $\mathbb{D}$ we get the Cartan decomposition:

$$G/K = KA^{++}, \quad G = KA^{+}K$$

If $g = k_1 a k_2$ then

$$g\,{}^t g = k_1 a^2 k_1^{-1}$$

so a^2 is the eigenvalue matrix of $g\,{}^t g$ and the columns of k_1 are its eigenvectors.

Here A^{++} is the group of diagonal matrices with first entry > 1, which can be arranged by choosing the eigenvalues in the correct order. I write $++$ rather than $+$ to take into account what happens for groups other than $\mathrm{SL}_2(\mathbb{R})$.

5. Vector fields

The action of a Lie group G on a manifold M determines also vector fields corresponding to vectors in its Lie algebra, the flows along the orbits of one-parameter subgroups $\exp(tX)$.

The element X in $\mathfrak{g}$ determines at m the vector

$$\frac{(I + \varepsilon X)m - m}{\varepsilon}$$

where we may assume $\varepsilon^2 = 0$.

Let's see what happens for

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

On $\mathcal{H}$:

$$\nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$$

$$z \longmapsto \frac{z+\varepsilon}{1} = z + \varepsilon$$

$$\nu_+ \longmapsto \frac{\partial}{\partial x}$$

On $\mathcal{H}$:

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{bmatrix}$$

$$\begin{aligned} z \longmapsto \frac{(1+\varepsilon)z}{(1-\varepsilon)} \\ &= z(1+\varepsilon)(1+\varepsilon+\varepsilon^2+\cdots) \\ &= z(1+2\varepsilon) = z + 2\varepsilon z \end{aligned}$$

$$\alpha \longmapsto 2x\,\frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial y}$$

On $\mathcal{H}$:

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{bmatrix}$$

$$z \longmapsto \frac{z - \varepsilon}{\varepsilon z + 1} = z - \varepsilon(1 + z^2)$$

$$\kappa \longmapsto -(1 + x^2 - y^2)\frac{\partial}{\partial x} - 2xy\,\frac{\partial}{\partial y}$$

On $\mathbb{D}$:

$$\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$$

$$z \longmapsto \frac{z + \varepsilon}{\varepsilon z + 1} = z + \varepsilon(1 - z^2)$$

$$\alpha \longmapsto (1 - z^2)$$

On $\mathbb{D}$:

$$\nu_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1-h & h \\ -h & 1+h \end{bmatrix} \quad (h = \varepsilon/2i)$$

$$z \longmapsto z + h(z-1)^2$$

$$\nu_+ \longmapsto (1/2i)(z-1)^2$$

6. Measures

Each of the decompositions or factorizations

$$\begin{aligned} G &= NAK && \big(\textbf{Iwasawa}\big) \\ &= P \cup PwN && \big(\textbf{Bruhat}\big) \\ &= KA^{++}K && \big(\textbf{Cartan}\big) \end{aligned}$$

corresponds to a different formula for integration on G**.**

$G = NAK$:

$$\int_G f(g)\,dg = \int_K dk \int_A \delta_P^{-1}(a)\,da \int_N f(nak)\,dn$$

This is because $G/K = \mathcal{H}$, $\mathcal{H} = P \cdot i$, and

$$\frac{1}{y} \cdot dx \cdot \frac{dy}{y}$$

is G-invariant.

We'll say more about this later on.

$G = P \cup PwN$:

$$\int_G f(g)\,dg = \int_N dn_2 \int_A \delta_P^{-1}(a)\,da \int_N f(n_1 a w n_2)\,dn_1$$

This will be explained later on, when we look at representations associated to the space $P\backslash G$.

$G = KA^{++}K$:

$$\int_G f(g)\,dg = \int_{K\times K} dk_1\,dk_2 \int_{A^{++}} |x^2 - x^{-2}| f(k_1 a_x k_2)\,da$$

Geometrically, this is equivalent to this assertion:

> *The circumference of the non-Euclidean circle in $\mathcal{H}$ through iy is $\pi|y - y^{-1}|$.*

This can be seen easily by transforming to $\mathbb{D}$. The image of iy is $(y-1)/y+1)$. On $\mathcal{H}$ $dy/y = dr$, and on $\mathbb{D}$ $dr = 2\,dt/(1-t^2)$. Then one can use radial symmetry to see that the non-Euclidean circumference at Euclidean radius t is $4\pi t/(1 - t^2)$, and interpret in terms of y.

7. Conjugation classes

Suppose g in SL_2. Its characteristic equation is

$$x^2 - \tau x + 1 = 0 \quad (\tau = \mathrm{trace}(g))$$

with roots

$$x = \frac{-\tau \pm \sqrt{\tau^2 - 4}}{2}$$

If $|\tau| > 2$ the roots are real and distinct and

$$g = X \begin{bmatrix} x_1 & \\ & x_2 \end{bmatrix} X^{-1}$$

fo some X in K. Since conjugation by the element

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

interchanges the order of diagonal entries, both x and x^{-1} give rise to the same conjugacy class.

$$x = \frac{-\tau \pm \sqrt{\tau^2 - 4}}{2}$$

If $|\tau| < 2$ the roots are complex, distinct, and of absolute value 1. If one is $c + is$ with $s > 0$ then

$$g = X \begin{bmatrix} c + is & 0 \\ 0 & c - is \end{bmatrix} X^{-1}$$

where

$$X = \begin{bmatrix} v & \overline{v} \end{bmatrix}$$

with v the eigenvector for $c + is$.

Then

$$g = X \begin{bmatrix} c & -s \\ s & c \end{bmatrix} X^{-1}$$

where now

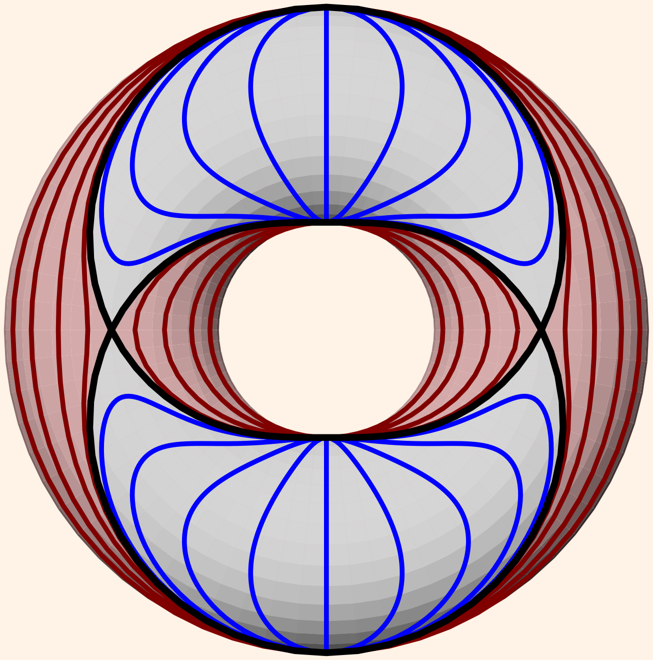
$$X = \begin{bmatrix} \mathrm{RE}\, v & \mathrm{IM}\, v \end{bmatrix}$$

If X has positive determinant then g is conjugate to the same rotation matrix in $\mathrm{SL}_2(\mathbb{R})$, but otherwise to its transpose (or inverse). Thus there is one class for each $0 < \theta < 2\pi$ excluding π. Geometrically, the question here is whether g rotates clockwise or counterclockwise.

If $|\tau| = \pm 2$ we get $\pm I$ and also 4 unipotent classes

$$\begin{bmatrix} \varepsilon & \pm 1 \\ 0 & \varepsilon \end{bmatrix} \quad (\varepsilon = \pm 1)$$

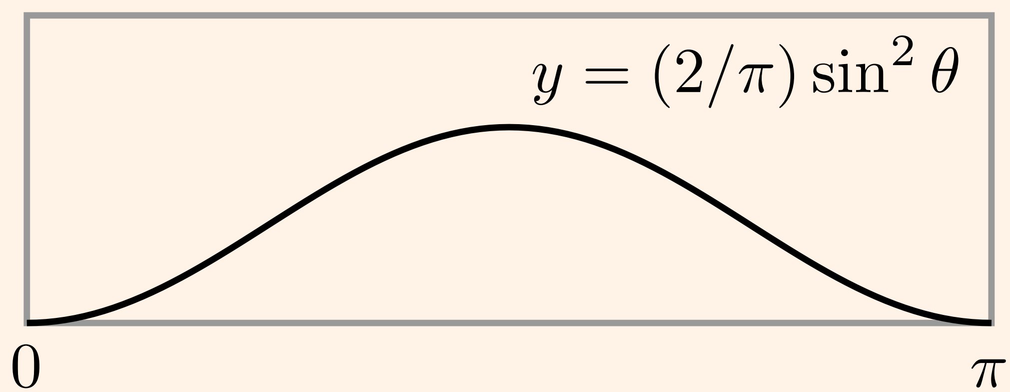
We can picture SL_2 as a solid torus, fibring by circles over the disk $\mathbb{D}$, and then partition it by conjugacy classes (elliptic, hyperbolic, unipotent):



Conjugation classes in the compact group $\mathrm{SU}(2)$ are simpler. Every g in G is conjugate to a unique diagonal matrix t in T with first entry $e^{i\theta}$, $0 \le \theta \le \pi$. Weyl's integration formula for $G = \mathrm{SU}(2)$ says

$$\int_G = \frac{1}{2} \int_{G/T} dx \int_T f(xtx^{-1}) \sin^2\theta \, dt$$

where measures are chosen so $G = (G/T) \times T$. The $1/2$ arises because in $SU(2)$ the order of eigenvalues doesn't matter. One thing the formula means is that if you choose a 2×2 unitary matrix with determinant 1 randomly you are more likely to get one with eigenvalues around i than around ± 1. In terms of density:



Something similar happens for SL_2, but taking into account the various eigenvalue possibilities. If T is either A or K, let G_T^{reg} be the open subset of g conjugate to regular elements of T, W_T the order of $N_G(T)/T$. Then

$$\int_{G_T^{\mathrm{reg}}} f(g)\,dg = \frac{1}{|W_T|}\int_T |D(t)|\,dt \int_{G/T} f(xtx^{-1})\,dx$$

where

$$D(t) = \det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{t}}(t) - I)$$

This is proved by looking at the differential of the conjugation map $G/T \times T \to G$.

For A

$$|D(a_x)| = |x^2 - 1|\,|x^{-2} - 1| = |x - x^{-1}|^2$$

while for K

$$|D(k_\theta)| = 4\sin^2\theta\,.$$

8. Lifting to the group

Since $\mathcal{H} = G/K$, functions on $\mathcal{H}$ may be lifted to functions on G invariant under right multiplication by K:

$$F(g) = f(g(i))\,.$$

It is often necessary to interpret vector fields on $\mathcal{H}$ in terms of the Lie algebra of G interpreted as left-invariant vector fields, acting on the right, on G.

The key to this translation process is a simple calculation:

$$[R_X F](g) = [L_{gXg^{-1}} F](g)$$

$$\textbf{since } F(g \cdot (I+\varepsilon)X) = F((I+\varepsilon \cdot gXg^{-1}) \cdot g)$$

$$[L_X F](g) = [R_{g^{-1}Xg} F](g)$$

On $\mathcal{H}$

$$p = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$$

takes i to $a^2 i + x$, so R_X as a vector field on $\mathcal{H}$ is $L_{pXp^{-1}}X$ where $a^2 = y$. This gives us:

$$\begin{aligned} R_\kappa &= 0 \\ R_\alpha &= 2y\,\partial/\partial y \\ R_{\nu_+} &= y\,\partial/\partial x \end{aligned}$$

There are some elements of the complex Lie algebra of G that are useful when dealing with the complex structure on $\mathcal{H}$. To motivate these, consider the adjoint representation of K on $\mathfrak{g}$. The subgroup K is a torus, and the Lie algebra breaks up first of all into skew-symmetric and symmetric matrices. The group K acts trivially on the anti-symmetric component, its won Lie algebra, and acts by rotation on the symmetric part, which may be identified with the tangent plane of $\mathcal{H}$ at i. The eigenvalues and eigenvectors are necessarily complex. To be precise, if

$$x_\pm = \begin{bmatrix} 0 & \mp i \\ \mp i & 0 \end{bmatrix}$$

then

$$k_\theta x_\pm k_\theta^{-1} = e^{\pm 2i\theta} x_\pm$$

Since $x_\pm = \alpha \mp i(\kappa + 2\nu_+)$, the previous formulas give us

$$\begin{aligned} R_{x_+} &= -2iy\,\partial/\partial z \\ R_{x_-} &= 2iy\,\partial/\partial \overline{z} \end{aligned}$$

What right action does the Laplacian correspond to?

There is a very special right-acting differential operator on G called the Casimir operator (to be explained in detail later):

$$C = \alpha^2/4 - \alpha/2 + \nu_+\nu_- = \alpha^2/4 - \alpha/2 + \nu_+\nu_+ + \nu_+\kappa$$

This satisfies

$$R_C = \Delta_{\mathcal{H}})$$

The End