IRREDUCIBLE HARISH CHANDRA MODULES FOR $\mathrm{SL}(2,\mathbb{R})$: REFERENCE GUIDE

label	condition	name	K-types	Ω action	infinitesimal character	leading exps.	unitary?	comments
$\mathbb{C}$		trivial	0	0	ρ	0	yes	unipotent
$F(d)$	$d \in \mathbb{Z}, d > 1$	d-dimensional	$d-1, \ldots, -(d-1)$	$\frac{1}{4}(d^2-1)$	$d\rho$	$\frac{1}{2}(-d+1)$	no	
D_n^+	$n \in \mathbb{Z}, n \geq 1$	holomorphic discrete series	$n+1, n+3, n+5, \ldots$	$\frac{1}{4}(n^2-1)$	$n\rho$	$\frac{1}{2}(n+1)$	yes	discrete series
D_n^-	$n \in \mathbb{Z}, n \geq 1$	anti-holomorphic discrete series	$-(n+1), -(n+3), \ldots$	$\frac{1}{4}(n^2-1)$	$n\rho$	$\frac{1}{2}(n+1)$	yes	discrete series
D_0^+		limit of discrete series	$1, 3, 5, \ldots$	$-\frac{1}{4}$	0	$\frac{1}{2}$	yes	tempered, unipotent
D_0^-		limit of discrete series	$-1, -3, -5, \ldots$	$-\frac{1}{4}$	0	$\frac{1}{2}$	yes	tempered, unipotent
$P_+(\nu)$	$\nu \in i\mathbb{R}^{\geq 0}$	spherical unitary principal series	$0, \pm 2, \pm 4, \ldots$	$\frac{1}{4}(\nu^2-1)$	$\nu\rho$	$\frac{1}{2}(\pm\nu+1)$	yes	tempered
$P_-(\nu)$	$\nu \in i\mathbb{R}^{>0}$	nonspherical unitary principal series	$\pm 1, \pm 3, \pm 5, \ldots$	$\frac{1}{4}(\nu^2-1)$	$\nu\rho$	$\frac{1}{2}(\pm\nu+1)$	yes	tempered
$P_+(\nu)$	$\nu \in (0,1)$	complementary series	$0, \pm 2, \pm 4, \ldots$	$\frac{1}{4}(\nu^2-1)$	$\nu\rho$	$\frac{1}{2}(\pm\nu+1)$	yes	Selberg's 1/4 Conjecture: not automorphic
$P_+(\nu)$	$\nu \in \mathbb{C}$: $\Re(\nu) > 0$, $\nu \notin (0,1)$, $\nu \neq 1, 3, \ldots$	principal series	$0, \pm 2, \pm 4, \ldots$	$\frac{1}{4}(\nu^2-1)$	$\nu\rho$	$\frac{1}{2}(\pm\nu+1)$	no	
$P_-(\nu)$	$\nu \in \mathbb{C}$: $\Re(\nu) > 0$, $\nu \neq 2, 4, \ldots$	principal series	$\pm 1, \pm 3, \pm 5, \ldots$	$\frac{1}{4}(\nu^2-1)$	$\nu\rho$	$\frac{1}{2}(\pm\nu+1)$	no	

Definition of principal series: $P_\varepsilon(\nu)$ is the Harish-Chandra module of the normalized induced representation $\mathrm{Ind}^\infty(\varepsilon \otimes e^\nu \otimes \mathbf{1}|P,G)$; so $P_\varepsilon(\nu) \simeq P_\varepsilon(-\nu)$ if $P_\varepsilon(\nu)$ is irreducible.
Canonical normalization of Casimir: $\Omega = \frac{1}{4}H^2 + \frac{1}{2}(XY+YX)$
Leading exponent: η a leading exponent means there is a matrix coefficient whose restriction to A grows like

$$\begin{pmatrix} a^{1/2} & 0 \\ 0 & a^{-1/2} \end{pmatrix} \mapsto a^\eta$$

as a tends to 0. Since $a \to 0$ corresponds to going to infinity in G, $\Re(\eta) > 0$ means "decay at infinity on G".

Reducibility of principal series. Fix a nonnegative integer n and set ε equal to the parity of n. We have the following exact sequences:

$$0 \longrightarrow D_n^- \oplus D_n^+ \longrightarrow P_\varepsilon(n) \longrightarrow F_n \longrightarrow 0;$$

and

$$0 \longrightarrow F_n \longrightarrow P_\varepsilon(-n) \longrightarrow D_n^- \oplus D_n^+ \longrightarrow 0.$$

Character Formulas:

Principal series. If $\pi = \mathrm{Ind}^\infty(\varepsilon \otimes e^\nu \otimes \mathbf{1}|P, G)$, the character is given by the following function:

$$\Theta_{\varepsilon,\nu}(g) = 0 \qquad \text{if } g \text{ is conjugate to } \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

and

$$\Theta_{\varepsilon,\nu}(g) = \frac{\varepsilon(e^{\nu t} + e^{-\nu t})}{|e^t - e^{-t}|} \qquad \text{if } g \text{ is conjugate to } \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Sums of discrete series. The character of $D_n^+ \oplus D_n^-$ (for $n \geq 1$) is given by:

$$\Theta_n(g) = \frac{-e^{in\theta} - e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} \qquad \text{if } g \text{ is conjugate to } \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

and

$$\Theta_n(g) = \epsilon^{n+1} \frac{2e^{-n|t|}}{|e^t - e^{-t}|} \qquad \text{if } g \text{ is conjugate to } \epsilon \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Individual discrete series. The character of $D_n^\pm$ $(n \geq 1)$ is given by:

$$\Theta_n^\pm(g) = \frac{\mp e^{-in\theta}}{e^{i\theta} - e^{-i\theta}} \qquad \text{if } g \text{ is conjugate to } \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};$$

and

$$\Theta_n^\pm(g) = \epsilon^{n+1} \frac{e^{-n|t|}}{|e^t - e^{-t}|} \qquad \text{if } g \text{ is conjugate to } \epsilon \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$