Application of Malliavin calculus to stochastic partial differential equations

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1 Introduction to Malliavin Calculus

The Malliavin calculus is an infinite dimensional calculus on a Gaussian space, which is mainly applied to establish the regularity of the law of nonlinear functionals of the underlying Gaussian process.

Suppose that H is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_H$. The norm of an element $h \in H$ will be denoted by $\|h\|_H$.

Consider a Gaussian family of random variables $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$, with zero mean and covariance

$$E(W(h)W(g)) = \langle h, g \rangle_H.$$

The mapping $h \rightarrow W(h)$ provides a linear isometry of H onto a closed subspace of $\mathcal{H}_1$ of $L^2(\Omega)$.

Example 1 If $B = \{B_t, t \geq 0\}$ is a Brownian motion, we take $H = L^2([0, \infty))$ and

$$W(h) = \int_0^\infty h(t) dB_t.$$

Example 2 In the case $H = L^2(T, \mathcal{B}, \mu)$, where μ is a σ-finite measure without atoms, for any set $A \in \mathcal{B}$ with $\mu(A) < \infty$ we define

$$W(A) = W(\mathbf{1}_A).$$

Then, $A \to W(A)$ is a Gaussian measure with independent increments (Gaussian white noise). That is, if $A_1, \ldots, A_n$ are disjoint sets with finite measure, the random variables $W(A_1), \ldots, W(A_n)$ are independent, and for any $A \in \mathcal{B}$ with $\mu(A) < \infty$, $W(A)$ has the distribution $N(0, \mu(A))$. In this case, any square integrable random variable $F \in L^2(\Omega, \mathcal{F}, P)$ (assuming that the σ-field $\mathcal{F}$ is generated by $\{W(h)\}$) admits the following Wiener chaos expansion

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n). \tag{1.1}$$

In this formula f_n is a symmetric function of $L^2(T^n)$ and I_n denotes the multiple stochastic integral introduced by Itô in [6]. In particular $I_1(f_1) = W(f_1)$. Furthermore,

$$E(F^2) = E(F)^2 + \sum_{n=1}^{\infty} n! \|f_n\|^2_{L^2(T^n)}.$$

1.1 Derivative operator

Let $\mathcal{S}$ denote the class of smooth and cylindrical random variables variable of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where f belongs to $C_p^\infty(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth order), $h_1, \ldots, h_n$ are in H, and $n \geq 1$.

The *derivative* of F is the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \ldots, W(h_n)) h_i.$$

Example $D(W(h)) = h$, $D(W(h)^2) = 2W(h)h$.

The following result is an integration by parts formula.

Proposition 1.1 *Suppose that F is a smooth and cylindrical random variable and $h \in H$. Then*

$$E(\langle DF, h\rangle_H) = E(FW(h)).$$

Proof. We can assume that there exist orthonormal elements of H, $e_1, \ldots, e_n$, such that $h = e_1$ and

$$F = f(W(e_1), \ldots, W(e_n)),$$

where $f \in C_p^\infty(\mathbb{R}^n)$. Let $\phi(x)$ denote the density of the standard normal distribution on $\mathbb{R}^n$, that is,

$$\phi(x) = (2\pi)^{-\frac{n}{2}} \exp(-\frac{1}{2}\sum_{i=1}^n x_i^2).$$

Then we have

$$\begin{aligned} E(\langle DF, h\rangle_H) &= E\left(\frac{\partial f}{\partial x_1}(W(e_1))\right) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x)\phi(x)dx \\ &= \int_{\mathbb{R}^n} f(x)\phi(x)x_1 dx = E(FW(e_1)), \end{aligned}$$

which completes the proof. ■

Applying the previous result to a product FG, we obtain the following consequence.

Proposition 1.2 *Suppose that F and G are smooth and cylindrical random variables, and $h \in H$. Then we have*

$$E(G\langle DF, h\rangle_H) = E(-F\langle DG, h\rangle_H + FGW(h)).$$

Proof. Use

$$D(FG) = FDG + GDF.$$

■

As a consequence we obtain the following result.

Proposition 1.3 *The operator D is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$ for any $p \geq 1$.*

Proof. Let $\{F_N, N \geq 1\}$ be a sequence of random variables in $\mathcal{S}$ such that

$$F_N \to 0 \text{ in } L^p(\Omega),$$

and

$$DF_N \to \eta \text{ in } L^p(\Omega; H),$$

as N tends to infinity. Then, we claim that $\eta = 0$. Indeed, for any $h \in H$ and for any random variable $F = f(W(h_1), \ldots, W(h_n)) \in \mathcal{S}$ such that f and its partial derivatives are bounded, and $FW(h)$ is bounded, we have

$$\begin{aligned} E(\langle \eta, h \rangle_H F) &= \lim_{N\to\infty} E(\langle DF_N, h \rangle_H F) \\ &= \lim_{N\to\infty} E(-F_N \langle DF, h \rangle_H + F_N F W(h)) = 0 \end{aligned}$$

This implies $\eta = 0$. ■

For any $p \geq 1$ we denote by $\mathbb{D}^{1,p}$ the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{1,p} = \left[E(|F|^p) + E(\|DF\|_H^p)\right]^{\frac{1}{p}}.$$

For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with the scalar product

$$\langle F, G \rangle_{1,2} = E(FG) + E(\langle DF, DG \rangle_H).$$

We can define the iteration of the operator D in such a way that for a random variable $F \in \mathcal{S}$, the iterated derivative $D^k F$ is a random variable with values in $H^{\otimes k}$.

For every $p \geq 1$ and any natural number $k \geq 1$ we introduce the seminorm on $\mathcal{S}$ defined by

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{j=1}^{k} E(\|D^j F\|_{H^{\otimes j}}^p)\right]^{\frac{1}{p}}.$$

We denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the seminorm $\|\cdot\|_{k,p}$.

For any $k \geq 1$ and $p > q$ we have $\mathbb{D}^{k,p} \subset \mathbb{D}^{k-1,q}$. We set $\mathbb{D}^\infty = \cap_{k,p} \mathbb{D}^{k,p}$.

Remark 1 If $H = \mathbb{R}^n$, then the spaces $\mathbb{D}^{k,p}$ can be identified as ordinary Sobolev spaces of functions on $\mathbb{R}^n$ that together with their k first partial derivatives have moments of order p with respect to the standard normal law.

Remark 2 The above definitions can be exended to Hilbert-valued random variables: $\mathbb{D}^{k,p}(V)$, where V is a given Hilbert space.

The following result is the chain rule, which can be easily proved by approximating the random variable F by smooth and cyllindrical random variables and the function φ by $\varphi * \psi_\epsilon$, where $\{\psi_\epsilon\}$ is an approximation of the identity.

Proposition 1.4 *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives, and fix $p \geq 1$. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Then $\varphi(F) \in \mathbb{D}^{1,p}$, and*

$$D(\varphi(F)) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F) DF^i.$$

The following Hölder inequality is proved easily, and it implies that $\mathbb{D}^\infty$ is closed by multiplication.

Proposition 1.5 *Let $F \in \mathbb{D}^{k,p}$, $G \in \mathbb{D}^{k,q}$ for $k \in \mathbb{N}^*$, $1 < p, q < \infty$ and let r be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then, $FG \in \mathbb{D}^{k,r}$ and*

$$\|FG\|_{k,r} \leq c_{p,q,k} \|F\|_{k,p} \|G\|_{k,q}.$$

Consider now the white noise case, that is, $H = L^2(T, \mathcal{B}, \mu)$. Then, the derivative DF is a random element in $L^2(\Omega; H) \sim L^2(\Omega \times T, \mathcal{F} \otimes \mathcal{B}, P \times \mu)$, that is, it is a stochastic process that we denote by $\{D_t F, t \in T\}$.

Suppose that F is a square integrable random variable having an orthogonal Wiener chaos expansion of the form

$$F = E(F) + \sum_{n=1}^{\infty} I_n(f_n), \tag{1.2}$$

where the kernels f_n are symmetric functions of $L^2(T^n)$. The derivative $D_t F$ can be easily computed using this expression.

Proposition 1.6 *Let $F \in L^2(\Omega)$ be a square integrable random variable with a development of the form (1.2). Then F belongs to $\mathbb{D}^{1,2}$ if and only if*

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2(T^n)} < \infty$$

and in this case we have

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)). \tag{1.3}$$

Proof. Suppose first that $F = I_n(f_n)$, where f_n is a symmetric and elementary function of the form

$$f(t_1,\dots,t_n) = \sum_{i_1,\dots,i_n=1}^{m} a_{i_1\cdots i_n}\mathbf{1}_{A_{i_1}\times\cdots\times A_{i_n}}(t_1,\dots,t_n), \tag{1.4}$$

where $A_1, A_2, \dots, A_m$ are pair-wise disjoint sets with finite measure, and the coefficients $a_{i_1\cdots i_n}$ are zero if any two of the indices $i_1,\dots,i_n$ are equal. Then

$$D_tF = \sum_{j=1}^{n}\sum_{i_1,\dots,i_n=1}^{m} a_{i_1\cdots i_n}W(A_{i_1})\cdots\mathbf{1}_{A_{i_j}}(t)\cdots W(A_{i_n}) = nI_{n-1}(f_n(\cdot,t)).$$

Then the result follows easily. ■

The heuristic meaning of the preceding proposition is clear. Suppose that F is a multiple stochastic integral of the form $I_n(f_n)$, which can be denoted by

$$I_n(f_n) = \int_T\cdots\int_T f_n(t_1,\dots,t_n)W(dt_1)\cdots W(dt_n).$$

Then, F belongs to the domain of the derivation operator and D_tF is obtained simply by removing one of the stochastic integrals, letting the variable t be free, and multiplying by the factor n.

We will make use of the following result.

Lemma 1.7 *Let $\{F_n, n\geq 1\}$ be a sequence of random variables converging to F in $L^p(\Omega)$ for some $p>1$. Suppose that $\sup_n \|F_n\|_{k,p} < \infty$ for some $k\geq 1$. Then Then F belongs to $\mathbb{D}^{k,p}$.*

Proof. We do the proof only in the case $p=2$, $k=1$ and assuming that we are in the white noise context. There exists a subsequence $\{F_{n(k)}, k\geq 1\}$ such that the sequence of derivatives $DF_{n(k)}$ converges in the weak topology of $L^2(\Omega\times T)$ to some element $\alpha\in L^2(\Omega\times T)$. Then, for any $h\in H$ the projections of $\langle h, DF_{n(k)}\rangle$ on any Wiener chaos converge in the weak topology of $L^2(\Omega)$, as k tends to infinity, to those of $\langle h,\alpha\rangle$. Consequently, Proposition 1.6 implies $F\in\mathbb{D}^{1,2}$ and $\alpha = DF$. Moreover, for any weakly convergent subsequence the limit must be equal to α by the preceding argument, and this implies the weak convergence of the whole sequence. ■

Proposition 1.8 *Let F be a random variable of the space $\mathbb{D}^{1,2}$ such that $DF = 0$. Then $F = E(F)$.*

Proof. In the white noise case, this proposition is obvious from the Wiener chaos expansion of the derivative provided in Proposition 1.6. In the general case the result is also true, even for random variables in $\mathbb{D}^{1,1}$. ■

Proposition 1.9 *Let $A \in \mathcal{F}$. Then the indicator function of A belongs to $\mathbb{D}^{1,2}$ if and only if $P(A)$ is equal to zero or one.*

Proof. By the chain rule (Proposition 1.4) applied to to a function $\varphi \in C_0^\infty(\mathbb{R})$, which is equal to x^2 on $[0,1]$, we have

$$D\mathbf{1}_A = D(\mathbf{1}_A)^2 = 2\mathbf{1}_A D\mathbf{1}_A$$

and, therefore, $D\mathbf{1}_A = 0$ because from the above equality we get that this derivative is zero on A^c and equal to twice its value on A. So, by the Proposition 1.8 we obtain $\mathbf{1}_A = P(A)$. ■

1.2 Divergence operator

We denote by δ the adjoint of the operator D (*divergence operator*). That is, δ is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ such that:

(i) The domain of δ, denoted by $\operatorname{Dom}\delta$, is the set of H-valued square integrable random variables $u \in L^2(\Omega; H)$ such that

$$|E(\langle DF, u\rangle_H)| \leq c\|F\|_2,$$

for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

(ii) If u belongs to $\operatorname{Dom}\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the duality relation

$$E(F\delta(u)) = E(\langle DF, u\rangle_H)$$

for any $F \in \mathbb{D}^{1,2}$.

Properties of the divergence

1. $E(\delta(u)) = 0$ (take $F = 1$ in the duality formula).

2. Consider the set $\mathcal{S}_H$ of H-valued cylindrical and smooth random variables of the form $u = \sum_{j=1}^n F_j h_j$, where the $F_j \in \mathcal{S}$, and $h_j \in H$. Then, Proposition 1.2 implies that an element $u \in \mathcal{S}_H$ belongs to the domain of δ and

$$\delta(u) = \sum_{j=1}^n F_j W(h_j) - \sum_{j=1}^n \langle DF_j, h_j \rangle_H .$$

We will make use of the notation $D_h F = \langle DF, h \rangle_H$, for any $h \in H$ and $F \in \mathbb{D}^{1,2}$.

Three basic formulas

Suppose that $u, v \in \mathcal{S}_H$, $F \in \mathcal{S}$ and $h \in H$. Then, if $\{e_i\}$ is a complete orthonormal system in H

$$E(\delta(u)\delta(v)) = E(\langle u, v \rangle_H) + E\left(\sum_{i,j=1}^\infty D_{e_i} \langle u, e_j \rangle_H D_{e_j} \langle v, e_i \rangle_H \right) \tag{1.5}$$

$$D_h(\delta(u)) = \delta(D_h u) + \langle h, u \rangle_H \tag{1.6}$$

$$\delta(Fu) = F\delta(u) - \langle DF, u \rangle_H . \tag{1.7}$$

Proof of (1.6): Assume $u = \sum_{j=1}^n F_j h_j$. Then

$$\begin{aligned} D_h(\delta(u)) &= D_h \left(\sum_{j=1}^n F_j W(h_j) - \sum_{j=1}^n \langle DF_j, h_j \rangle_H \right) \\ &= \sum_{j=1}^n F_j \langle h, h_j \rangle_H + \sum_{j=1}^n \big(D_h F_j W(h_j) - \langle D_h (DF_j), h_j \rangle_H \big) \\ &= \langle u, h \rangle_H + \delta(D_h u). \end{aligned}$$

Proof of (1.5): Using the duality formula and (1.6) yields

$$\begin{aligned}
E(\delta(u)\delta(v)) &= E\left(\langle v, D(\delta(u))\rangle_H\right) = E\left(\sum_{i=1}^{\infty}\langle v, e_i\rangle_H D_{e_i}(\delta(u))\right) \\
&= E\left(\sum_{i=1}^{\infty}\langle v, e_i\rangle_H \ \left(\langle u, e_i\rangle_H + \delta(D_{e_i}u)\right)\right) \\
&= E\left(\langle u, v\rangle_H\right) + E\left(\sum_{i,j=1}^{\infty} D_{e_i}\langle u, e_j\rangle_H \ D_{e_j}\langle v, e_i\rangle_H\right).
\end{aligned}$$

Proof of (1.7): For any smooth random variable $G \in \mathcal{S}$ we have

$$\begin{aligned}
E\left(\langle DG, Fu\rangle_H\right) &= E\left(\langle u, D(FG) - GDF\rangle_H\right) \\
&= E\left(\left(\delta(u)F - \langle u, DF\rangle_H\right) G\right).
\end{aligned}$$

Remark 1 Property (1.5) implies the estimate

$$E\left(\delta(u)^2\right) \le E\left(\|u\|_H^2\right) + E\left(\|Du\|_{H\otimes H}^2\right) = \|u\|_{1,2,H}^2 .$$

As a consequence, $\mathbb{D}^{1,2}(H) \subset \operatorname{Dom}\delta$.

Remark 2 Properties (1.5), (1.6) and (1.7) hold under more general conditions:

1. $u \in \mathbb{D}^{1,2}(H)$ for (1.5).
2. $u \in \mathbb{D}^{1,2}(H)$ and $D_h u$ belongs to $\operatorname{Dom}\delta$ for (1.6).
3. $F \in \mathbb{D}^{1,2}$, $u \in \operatorname{Dom}\delta$, $Fu \in L^2(\Omega; H)$, and $F\delta(u) - \langle DF, u\rangle_H \in L^2(\Omega)$ for (1.7).

Consider the case of a Gaussian white noise $H = L^2(T, \mathcal{B}, \mu)$.

Proposition 1.10 *Fix a set $A \in \mathcal{B}$ with finite measure. Let $\mathcal{F}_{A^c}$ be the σ-field generated by the random variables $\{W(B), B \subset A^c\}$. Suppose that $F \in L^2(\Omega, \mathcal{F}_{A^c}, P)$. Then $F\mathbf{1}_A$ belongs to the domain of the divergence and*

$$\delta(F\mathbf{1}_A) = FW(A).$$

Proof. If F is cyllindrical and smooth, then

$$\delta(F\mathbf{1}_A) = FW(A) - \langle DF, \mathbf{1}_A \rangle_H = FW(A) - \int_A D_t F \mu(dt) = FW(A),$$

because $D_t F = 0$ if $t \in A$. ■

Consider the particular case $T = [0, \infty)$. Then $B_t = W(\mathbf{1}_{[0,t]})$ is a Brownian motion. Let $\mathcal{F}_t$ be the σ-field generated by the random variables $\{B_s, 0 \leq s \leq t\}$. We say that a stochastic process $\{u_t, t \geq 0\}$ is adapted if for all $t \geq 0$ the random variable u_t is $\mathcal{F}_t$ measurable.Then, the class L^2_a of adapted stochastic processes such that $E\left(\int_0^\infty u_t^2 dt\right) < \infty$ is included in the domain of the divergence and $\delta(u)$ coincides with the Itô stochastic integral:

$$\delta(u) = \int_0^\infty u_t dB_t.$$

This is a consequence of Proposition 1.10 and the fact that the operator δ is closed.

The following theorem is based on Meyer inequalities and it is a central result in Malliavin Calculus:

Theorem 1.11 *The operator δ is continuous from $\mathbb{D}^{k,p}(H)$ into $\mathbb{D}^{k-1,p}$ for all $p > 1$ and $k \geq 1$. That is,*

$$\|\delta(u)\|_{k-1,p} \leq C_{k,p} \|u\|_{k,p}.$$

The following proposition provides a precise estimate for the norm p of the divergence operator.

Proposition 1.12 *Let u be an element of $\mathbb{D}^{1,p}(H)$, $p > 1$. Then we have*

$$\|\delta(u)\|_p \leq c_p \left(\|E(u)\|_H + \|Du\|_{L^p(\Omega; H \otimes H)} \right).$$

Exercises

1.1 Let $F \in \mathbb{D}^{k,2}$ be given by the expansion $F = E(F) + \sum_{n=1}^\infty I_n(f_n)$. Show that for all $k \geq 1$,

$$D^k_{t_1,\ldots,t_k} F = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) I_{n-k}(f_n(\cdot, t_1, \ldots, t_k)),$$

and

$$E(\|D^k F\|^2_{L^2(T^k)}) = \sum_{n=k}^{\infty} \frac{n!^2}{(n-k)!} \|f_n\|^2_{L^2(T^n)}.$$

1.2 Suppose that $F = E(F) + \sum_{n=1}^{\infty} I_n(f_n)$ is a random variable belonging to the space $\mathbb{D}^{\infty,2} = \cap_k \mathbb{D}^{k,2}$. Show that $f_n = \frac{1}{n!} E(D^n F)$ for every $n \geq 1$ (Stroock's formula).

1.3 Let $F = \exp(W(h) - \frac{1}{2}\int_T h_s^2 \mu(ds))$, $h \in L^2(T)$. Compute the iterated derivatives of F and the kernels of its expansion into the Wiener chaos.

1.4 Let $F \in \mathbb{D}^{1,2}$ be a random variable such that $E(|F|^{-2}) < \infty$. Then $P\{F > 0\}$ is zero or one.

1.5 Suppose that $H = L^2(T)$. Let δ^k be the adjoint of the operator D^k. That is, a multiparameter process $u \in L^2(\Omega \times T^k)$ belongs to the domain of δ^k if and only if there exists a random variable $\delta^k(u)$ such that

$$E(F\delta^k(u)) = E(\langle u, D^k F\rangle_{L^2(T^k)})$$

for all $F \in \mathbb{D}^{k,2}$. Show that a process $u \in L^2(\Omega \times T^k)$ with an expansion

$$u_t = E(u_t) + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t)), \quad t \in T^k,$$

belongs to the domain of δ^k if and only if the series

$$\delta^k(u) = \int_T E(u_t) dW_t + \sum_{n=1}^{\infty} I_{n+k}(f_n)$$

converges in $L^2(\Omega)$.

1.6 Let $\{W_t, t \in [0,1]\}$ be a one-dimensional Brownian motion. Using Exercise 1.2 find the Wiener chaos expansion of the random variables

$$F_1 = \int_0^1 (t^3 W_t^3 + 2tW_t^2) dW_t, \quad F_2 = \int_0^1 t e^{W_t} dW_t.$$

2 Application of Malliavin Calculus to regularity of probability laws

The integration-by-parts formula leads to the following explicit expression for the density of a one-dimensional random variable.

Proposition 2.1 *Let F be a random variable in the space $\mathbb{D}^{1,2}$. Suppose that $\frac{DF}{\|DF\|_H^2}$ belongs to the domain of the operator δ in $L^2(\Omega)$. Then the law of F has a continuous and bounded density given by*

$$p(x) = E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right]. \tag{2.8}$$

Proof. Let ψ be a nonnegative smooth function with compact support, and set $\varphi(y) = \int_{-\infty}^{y}\psi(z)dz$. We know that $\varphi(F)$ belongs to $\mathbb{D}^{1,2}$, and making the scalar product of its derivative with DF obtains

$$\langle D(\varphi(F)), DF\rangle_H = \psi(F)\|DF\|_H^2.$$

Using the duality formula we obtain

$$\begin{aligned} E[\psi(F)] &= E\left[\left\langle D(\varphi(F)), \frac{DF}{\|DF\|_H^2}\right\rangle_H\right] \\ &= E\left[\varphi(F)\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right]. \end{aligned} \tag{2.9}$$

By an approximation argument, Equation (2.9) holds for $\psi(y) = \mathbf{1}_{[a,b]}(y)$, where $a < b$. As a consequence, we apply Fubini's theorem to get

$$\begin{aligned} P(a \le F \le b) &= E\left[\left(\int_{-\infty}^{F}\psi(x)dx\right)\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right] \\ &= \int_a^b E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right]dx, \end{aligned}$$

which implies the desired result. ∎

Notice that Equation (2.8) still holds under the hypotheses $F \in \mathbb{D}^{1,p}$ and $\frac{DF}{\|DF\|_H^2} \in \mathbb{D}^{1,p'}(H)$ for some $p, p' > 1$.

From expression (2.8) we can deduce estimates for the density. Fix p and q such that $\frac{1}{p}+\frac{1}{q}=1$. By Hölder's inequality we obtain

$$p(x) \leq (P(F>x))^{1/q} \left\| \delta\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_p .$$

In the same way, taking into account the relation $E[\delta(DF/\|DF\|_H^2)]=0$ we can deduce the inequality

$$p(x) \leq (P(F<x))^{1/q} \left\| \delta\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_p .$$

As a consequence, we obtain

$$p(x) \leq (P(|F|>|x|))^{1/q} \left\| \delta\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_p , \tag{2.10}$$

for all $x \in \mathbb{R}$. Now using the $L^p(\Omega)$ estimate of the operator δ established in Proposition 1.12 we obtain

$$\left\| \delta\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_p \leq c_p \left(\left\| E\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_H + \left\| D\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_{L^p(\Omega;H\otimes H)} \right). \tag{2.11}$$

We have

$$D\left(\frac{DF}{\|DF\|_H^2}\right) = \frac{D^2F}{\|DF\|_H^2} - 2\frac{\langle D^2F, DF\otimes DF\rangle_{H\otimes H}}{\|DF\|_H^4},$$

and, hence,

$$\left\| D\left(\frac{DF}{\|DF\|_H^2}\right)\right\|_{H\otimes H} \leq \frac{3\left\|D^2F\right\|_{H\otimes H}}{\|DF\|_H^2}. \tag{2.12}$$

Finally, from the inequalities (2.10), (2.11) and (2.12) we deduce the following estimate.

Proposition 2.2 *Let q, α, β be three positive real numbers such that $\frac{1}{q}+\frac{1}{\alpha}+\frac{1}{\beta}=1$. Let F be a random variable in the space $\mathbb{D}^{2,\alpha}$, such that $E(\|DF\|_H^{-2\beta})<\infty$. Then the density $p(x)$ of F can be estimated as follows*

$$\begin{aligned} p(x) &\leq c_{q,\alpha,\beta}\,(P(|F|>|x|))^{1/q} \\ &\quad \times \left(E(\|DF\|_H^{-1}) + \left\|D^2F\right\|_{L^\alpha(\Omega;H\otimes H)} \left\| \|DF\|_H^{-2}\right\|_\beta \right). \end{aligned} \tag{2.13}$$

Suppose that $F = (F^1, \dots, F^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,1}$. We associate to F the following random symmetric nonnegative definite matrix:

$$\gamma_F = (\langle DF^i, DF^j \rangle_H)_{1 \le i,j \le m}.$$

This matrix will be called the *Malliavin matrix* of the random vector F. The basic condition for the absolute continuity of the law of F will be that the matrix γ_F is invertible a.s. In this sense we have the following result, proved by Bouleau and Hirsch (see[1]):

Theorem 2.3 *Let $F = (F^1, \dots, F^m)$ be a random vector verifying the following conditions:*

(i) $F^i \in \mathbb{D}^{1,2}$ *for all* $i = 1, \dots, m$ *.*

(ii) The matrix γ_F *safistifes* $\det \gamma_F > 0$ *almost surely.*

Then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.

Condition (i) in Theorem 2.3 implies that the measure $(\det(\gamma_F) \cdot P) \circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$. In other words, the random vector F has an absolutely continuous law conditioned by the set $\{\det(\gamma_F) > 0\}$; that is,

$$P\{F \in B, \det(\gamma_F) > 0\} = 0$$

for any Borel subset B of $\mathbb{R}^m$ of zero Lebesgue measure.

The following theorem is the general criterion for the smoothness of densities.

Theorem 2.4 *Let $F = (F^1, \dots, F^m)$ be a random vector verifying the following conditions:*

(i) $F^i \in \mathbb{D}^\infty$ *for all* $i = 1, \dots, m$*.*

(ii) The matrix γ_F *safistifes* $E[(\det \gamma_F)^{-p}] < \infty$ *for all* $p \ge 2$*.*

Then the law of F possesses an infinitely differentiable density.

A random vector satisfying the conditions of Theorem 2.4 is called *nondegenerate*. For any multiindex $\alpha \in \{1, \ldots, m\}^k$, $k \geq 1$ we will denote by ∂_α the partial derivative $\frac{\partial^k}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$.

For the proof of Theorem 2.4 we need some preliminary material.

Lemma 2.5 *Suppose that γ is an $m \times m$ random matrix that is invertible a.s. and such that $|\det \gamma|^{-1} \in L^p(\Omega)$ for all $p \geq 1$. Suppose that the entries γ^{ij} of γ are in $\mathbb{D}^\infty$. Then $(\gamma^{-1})^{ij}$ belongs to $\mathbb{D}^\infty$ for all i, j, and*

$$D\left(\gamma^{-1}\right)^{ij} = -\sum_{k,l=1}^{m} \left(\gamma^{-1}\right)^{ik} \left(\gamma^{-1}\right)^{lj} D\gamma^{kl}. \tag{2.14}$$

Proof. First notice that $\{\det \gamma > 0\}$ has probability zero or one (see Exercise 1.4). We will assume that $\det \gamma > 0$ a.s. For any $\epsilon > 0$ define

$$\gamma_\epsilon^{-1} = \frac{\det \gamma}{\det \gamma + \epsilon} \gamma^{-1}.$$

Note that $(\det \gamma + \epsilon)^{-1}$ belongs to $\mathbb{D}^\infty$ because it can be expressed as the composition of $\det \gamma$ with a function in $C_p^\infty(\mathbb{R})$. Therefore, the entries of γ_ϵ^{-1} belong to $\mathbb{D}^\infty$. Furthermore, for any i, j, $(\gamma_\epsilon^{-1})^{ij}$ converges in $L^p(\Omega)$ to $(\gamma^{-1})^{ij}$ as ϵ tends to zero. Then, in order to check that the entries of γ^{-1} belong to $\mathbb{D}^\infty$, it suffices to show (taking into account Lemma 1.7) that the iterated derivatives of $(\gamma_\epsilon^{-1})^{ij}$ are bounded in $L^p(\Omega)$, uniformly with respect to ϵ, for any $p \geq 1$. This boundedness in $L^p(\Omega)$ holds, from the Leibnitz rule for the operator D^k, because $(\det \gamma)\gamma^{-1}$ belongs to $\mathbb{D}^\infty$, and on the other hand, $(\det \gamma + \epsilon)^{-1}$ has bounded $\|\cdot\|_{k,p}$ norms for all k, p, due to our hypotheses.

Finally, from the expression $\gamma_\epsilon^{-1}\gamma = \frac{\det \gamma}{\det \gamma + \epsilon} I$, we deduce Eq. (2.14) by first applying the derivative operator D and then letting ϵ tend to zero. ∎

Proposition 2.6 *Let $F = (F^1, \ldots, F^m)$ be a nondegenerate random vector. Let $G \in \mathbb{D}^\infty$ and let φ be a function in the space $C_p^\infty(\mathbb{R}^m)$. Then for any multiindex $\alpha \in \{1, \ldots, m\}^k$, $k \geq 1$, there exists an element $H_\alpha(F, G) \in \mathbb{D}^\infty$ such that*

$$E\left[\partial_\alpha \varphi(F) G\right] = E\left[\varphi(F) H_\alpha(F, G)\right], \tag{2.15}$$

where the elements $H_\alpha(F, G)$ are recursively given by

$$H_{(i)}(F, G) = \sum_{j=1}^{m} \delta\left(G\left(\gamma_F^{-1}\right)^{ij} DF^j\right), \tag{2.16}$$

$$H_\alpha(F, G) = H_{\alpha_k}(F, H_{(\alpha_1, \ldots, \alpha_{k-1})}(F, G)). \tag{2.17}$$

Proof. By the chain rule (Proposition 1.4) we have

$$\langle D(\varphi(F)), DF^j\rangle_H = \sum_{i=1}^m \partial_i\varphi(F)\langle DF^i, DF^j\rangle_H = \sum_{i=1}^m \partial_i\varphi(F)\gamma_F^{ij},$$

and, consequently,

$$\partial_i\varphi(F) = \sum_{j=1}^m \langle D(\varphi(F)), DF^j\rangle_H (\gamma_F^{-1})^{ji}.$$

Taking expectations and using the duality relationship between the derivative and the divergence operators we get

$$E\left[\partial_i\varphi(F)G\right] = E\left[\varphi(F)H_{(i)}(F,G)\right],$$

where $H_{(i)}$ equals to the right-hand side of Equation (2.16). Equation (2.17) follows by recurrence. Notice that the continuity of the operator δ (Theorem 1.11), and Lemma 2.5 imply that $H_{(i)}$ belongs to $\mathbb{D}^\infty$. Equation (2.17) follows by recurrence. ■

As a consequence, there exists constants β, $\gamma > 1$ and integers n, m such that

$$\|H_\alpha(F,G)\|_p \le c_{p,q}\left\|\det\gamma_F^{-1}\right\|_\beta^m \|DF\|_{k,\gamma}^n \|G\|_{k,q}.$$

Proof of Theorem 2.4. Equality (2.15) applied to the multiindex $\alpha = (1,2,\ldots,m)$ yields

$$E\left[G\partial_\alpha\varphi(F)\right] = E[\varphi(F)H_\alpha(F,G)].$$

Notice that

$$\varphi(F) = \int_{-\infty}^{F^1}\cdots\int_{-\infty}^{F^m}\partial_\alpha\varphi(x)dx.$$

Hence, by Fubini's theorem we can write

$$E\left[G\partial_\alpha\varphi(F)\right] = \int_{\mathbb{R}^m}\partial_\alpha\varphi(x)E\left[\mathbf{1}_{\{F>x\}}H_\alpha(F,G)\right]dx. \tag{2.18}$$

We can take as $\partial_\alpha\varphi$ any function in $C_0^\infty(\mathbb{R}^m)$. Then Equation (2.18) implies that the random vector F has a density given by

$$p(x) = E\left[\mathbf{1}_{\{F>x\}}H_\alpha(F,1)\right].$$

Moreover, for any multiindex β we have

$$\begin{aligned} E\left[\partial_\beta \partial_\alpha \varphi(F)\right] &= E[\varphi(F) H_\beta(F, H_\alpha(F,1)))] \\ &= \int_{\mathbb{R}^m} \partial_\alpha \varphi(x) E\left[\mathbf{1}_{\{F>x\}} H_\beta(H_\alpha)\right] dx. \end{aligned}$$

Hence, for any $\xi \in C_0^\infty(\mathbb{R}^m)$

$$\int_{\mathbb{R}^m} \partial_\beta \xi(x) p(x) dx = \int_{\mathbb{R}^m} \xi(x) E\left[\mathbf{1}_{\{F>x\}} H_\beta(F, H_\alpha(F,1))\right] dx.$$

Therefore $p(x)$ is infinitely differentiable, and for any multiindex β we have

$$\partial_\beta p(x) = (-1)^{|\beta|} E\left[\mathbf{1}_{\{F>x\}} H_{\beta(F,}(H_\alpha(F,1))\right].$$

■

2.1 Properties of the support of the law

Given a random vector $F : \Omega \to \mathbb{R}^m$, the topological support of the law of F is defined as the set of points $x \in \mathbb{R}^m$ such that $P(|x - F| < \varepsilon) > 0$ for all $\varepsilon > 0$. The following result asserts the connectivity property of the support of a smooth random vector.

Proposition 2.7 *Let $F = (F^1, \ldots, F^m)$ be a random vector whose components belong to $\mathbb{D}^{1,2}$. Then, the topological support of the law of F is a closed connected subset of $\mathbb{R}^m$.*

Proof. If the support of F is not connected, it can be decomposed as the union of two nonempty disjoint closed sets A and B.

For each integer $M \geq 2$ let $\psi_M : \mathbb{R}^m \to \mathbb{R}$ be an infinitely differentiable function such that $0 \leq \psi_M \leq 1$, $\psi_M(x) = 0$ if $|x| \geq M$, $\psi_M(x) = 1$ if $|x| \leq M - 1$, and $\sup_{x,M} |\nabla \psi_M(x)| < \infty$.

Set $A_M = A \cap \{|x| \leq M\}$ and $B_M = B \cap \{|x| \leq M\}$. For M large enough we have $A_M \neq \emptyset$ and $B_M \neq \emptyset$, and there exists an infinitely differentiable function f_M such that $0 \leq f_M \leq 1$, $f_M = 1$ in a neighborhood of A_M, and $f_M = 0$ in a neighborhood of B_M.

The sequence $(f_M\psi_M)(F)$ converges a.s. and in $L^2(\Omega)$ to $\mathbf{1}_{\{F\in A\}}$ as M tends to infinity. On the other hand, we have

$$\begin{aligned} D\left[(f_M\psi_M)(F)\right] &= \sum_{i=1}^m \left[(\psi_M\partial_i f_M)(F)DF^i + (f_M\partial_i\psi_M)(F)DF^i\right] \\ &= \sum_{i=1}^m (f_M\partial_i\psi_M)(F)DF^i. \end{aligned}$$

Hence,

$$\sup_M \|D\left[(f_M\psi_M)(F)\right]\|_H \le \sum_{i=1}^m \sup_M \|\partial_i\psi_M\|_\infty \left\|DF^i\right\|_H \in L^2(\Omega).$$

By Lemma 1.7 we get that $\mathbf{1}_{\{F\in A\}}$ belongs to $\mathbb{D}^{1,2}$, and by Proposition 1.9 this is contradictory because $0 < P(F \in A) < 1$. ■

As a consequence, the support of the law of a random variable $F \in \mathbb{D}^{1,2}$, is a closed interval. The next result provides sufficient conditions for the density of F to be nonzero in the interior of the support.

Proposition 2.8 *Let $F \in \mathbb{D}^{1,p}$, $p > 2$, and suppose that F possesses a density $p(x)$ which is locally Lipschitz in the interior of the support of the law of F. Let a be a point in the interior of the support of the law of F. Then $p(a) > 0$.*

Proof. Suppose $p(a) = 0$. Set $r = \frac{2p}{p+2} > 1$. From Proposition 1.9 we know that $\mathbf{1}_{\{F>a\}} \notin \mathbb{D}^{1,r}$ because $0 < P(F > a) < 1$. Fix $\epsilon > 0$ and set

$$\varphi_\epsilon(x) = \int_{-\infty}^x \frac{1}{2\epsilon}\mathbf{1}_{[a-\epsilon,a+\epsilon]}(y)dy.$$

Then $\varphi_\epsilon(F)$ converges to $\mathbf{1}_{\{F>a\}}$ in $L^r(\Omega)$ as $\epsilon \downarrow 0$. Moreover, $\varphi_\epsilon(F) \in \mathbb{D}^{1,r}$ and

$$D(\varphi_\epsilon(F)) = \frac{1}{2\epsilon}\mathbf{1}_{[a-\epsilon,a+\epsilon]}(F)DF.$$

We have

$$E\left(\|D(\varphi_\epsilon(F))\|_H^r\right) \le \left(E(\|DF\|_H^p)\right)^{\frac{2}{p+2}} \left(\frac{1}{(2\epsilon)^2}\int_{a-\epsilon}^{a+\epsilon} p(x)dx\right)^{\frac{p}{p+2}}.$$

The local Lipschitz property of p implies that $p(x) \leq K|x-a|$, and we obtain

$$E\left(\|D(\varphi_\epsilon(F))\|_H^r\right) \leq \left(E(\|DF\|_H^p\right)^{\frac{2}{p+2}} 2^{-r} K^{\frac{p}{p+2}}.$$

By Lemma 1.7 this implies $\mathbf{1}_{\{F>a\}} \in \mathbb{D}^{1,r}$, resulting in a contradiction. ■

The following example shows that, unlike the one-dimensional case, in dimension $m > 1$ the density of a nondegenerate random vector may vanish in the interior of the support.

Example Let h_1 and h_2 be two orthonormal elements of H. Define $X = (X_1, X_2)$, $X_1 = \arctan W(h_1)$, and $X_2 = \arctan W(h_2)$. Then, $X_i \in \mathbb{D}^\infty$ and

$$DX_i = (1 + W(h_i)^2)^{-1} h_i,$$

for $i = 1, 2$, and

$$\det \gamma_X = \left[(1 + W(h_1)^2)(1 + W(h_2)^2)\right]^{-2}.$$

The support of the law of the random vector X is the rectangle $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2$, and the density of X is strictly positive in the interior of the support. Now consider the vector $Y = (Y_1, Y_2)$ given by

$$\begin{aligned} Y_1 &= (X_1 + \frac{3\pi}{2}) \cos(2X_2 + \pi), \\ Y_2 &= (X_1 + \frac{3\pi}{2}) \sin(2X_2 + \pi). \end{aligned}$$

We have that $Y_i \in \mathbb{D}^\infty$ for $i = 1, 2$, and

$$\det \gamma_Y = 4(X_1 + \frac{3\pi}{2})^2 \left[(1 + W(h_1)^2)(1 + W(h_2)^2)\right]^{-2}.$$

This implies that Y is a nondegenerate random vector. Its support is the set $\{(x, y) : \pi^2 \leq x^2 + y^2 \leq 4\pi^2\}$, and the density of Y vanishes on the points (x, y) in the support such that $\pi < y < 2\pi$ and $x = 0$.

Exercises

2.1 Show that if F is a random variable in $\mathbb{D}^{2,4}$ such that $E(\|DF\|^{-8}) < \infty$, then $\frac{DF}{\|DF\|^2} \in \operatorname{Dom} \delta$ and

$$\delta\left(\frac{DF}{\|DF\|_H^2}\right) = \frac{\delta(DF)}{\|DF\|_H^2} - 2\frac{\langle DF \otimes DF, D^2 F\rangle_{H \otimes H}}{\|DF\|_H^4}.$$

2.2 Let F be a random variable in $\mathbb{D}^{1,2}$ such that $G_k \frac{DF}{\|DF\|_H^2}$ belongs to $\operatorname{Dom}\delta$ for any $k = 0, \dots, n$, where $G_0 = 1$ and

$$G_k = \delta \left(G_{k-1} \frac{DF}{\|DF\|_H^2} \right)$$

if $1 \le k \le n+1$. Show that F has a density of class C^n and

$$f^{(k)}(x) = (-1)^k E\left[\mathbf{1}_{\{F>x\}} G_{k+1} \right],$$

$0 \le k \le n$.

2.3 Set $M_t = \int_0^t u(s) dW_s$, where $W = \{W(t), t \in [0,T]\}$ is a Brownian motion and let $u = \{u(t), t \in [0,T]\}$ is an adapted process such that $|u(t)| \ge \rho > 0$ for some constant ρ, $E\left(\int_0^T u(t)^2 dt\right) < \infty$, $u(t) \in \mathbb{D}^{2,2}$ for each $t \in [0,T]$, and

$$\lambda := \sup_{s,t \in [0,T]} E(|D_s u_t|^p) + \sup_{r,s \in [0,T]} E((\int_0^T |D^2_{r,s} u_t|^p dt)^{\frac{p}{2}}) < \infty,$$

for some $p > 3$. Show that the density of M_t, denoted by $p_t(x)$ satisfies

$$p_t(x) \le \frac{c}{\sqrt{t}} P(|M_t| > |x|)^{\frac{1}{q}},$$

for all $t > 0$, where $q > \frac{p}{p-3}$ and the constant c depends on λ, $\rho$and p.

2.4 Let $F \in \mathbb{D}^{3,\alpha}$, $\alpha > 4$, be a random variable such that $E(\|DF\|_H^{-p}) < \infty$ for all $p \ge 2$. Show that the density $p(x)$ of F is continuously differentiable, and compute $p'(x)$.

2.5 Show that the random variable $F = \int_0^1 t^2 \arctan(W_t) dt$, where W is a Brownian motion, has a C^∞ density.

2.6 Let $W = \{W(s,t), (s,t) \in [0,1]^2\}$ be a two-parameter Wiener process. Show that $\sup_{(s,t)\in[0,1]^2} W(s,t)$ has an absolutely continuous distribution. Show also that the density is strictly positive in $(0, +\infty)$.

3 Stochastic heat equation

Suppose that $W = \{W(A), A \in \mathcal{B}(\mathbb{R}^2), |A| < \infty\}$ is a Gaussian family of random variables with zero mean and covariance

$$E(W(A)W(B) = |A \cap B|.$$

That is, W is a Gaussian white noise on the plane. Then if we set $W(t,x) = W([0,t] \times [0,x])$, for $t, x \geq 0$, $W = \{W(t,x), t \in [0,\infty), x \in [0,\infty)\}$ is a two-parameter Wiener process.

For each $t \geq 0$ we will denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,x), s \in [0,t], x \geq 0\}$ and the P-null sets. We say that a random field $\{u(t,x), t \geq 0, x \geq 0, \infty)\}$ is adapted if for all (t,x) the random variable $u(t,x)$ is $\mathcal{F}_t$-measurable.

Consider the following parabolic stochastic partial differential equation on $[0,\infty) \times [0,1]$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b(u(t,x)) + \sigma(u(t,x)) \frac{\partial^2 W}{\partial t \partial x} \tag{3.19}$$

with initial condition $u(0,x) = u_0(x)$, and Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$. We will assume that $u_0 \in C([0,1])$ satisfies $u_0(0) = u_0(1) = 0$.

Equation (3.19) is formal because the derivative $\frac{\partial^2 W}{\partial t \partial x}$ does not exist, and we will replace it by the following integral equation:

$$\begin{aligned} u(t,x) &= \int_0^1 G_t(x,y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x,y) b(u(s,y)) dy ds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y) \sigma(u(s,y)) W(dy,ds) \ , \end{aligned} \tag{3.20}$$

where $G_t(x,y)$ is the fundamental solution of the heat equation on $[0,1]$ with Dirichlet boundary conditions:

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial y^2}, \quad G_0(x,y) = \delta_x(y).$$

The kernel $G_t(x,y)$ has the following explicit formula:

$$\begin{aligned} G_t(x,y) &= \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left(-\frac{(y-x-2n)^2}{4t} \right) \right. \\ &\quad \left. - \exp\left(-\frac{(y+x-2n)^2}{4t} \right) \right\}. \end{aligned} \tag{3.21}$$

On the other hand, $G_t(x,y)$ coincides with the probability density at point y of a Brownian motion with variance $2t$ starting at x and killed if it leaves the interval $[0,1]$:

$$G_t(x,y) = \frac{d}{dy} E^x\{B_t \in dy, B_s \in (0,1) \quad \forall s \in [0,t]\}$$

This implies that

$$G_t(x,y) \leq \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{3.22}$$

Therefore, for any $\beta > 0$ we have

$$\int_0^1 G_t(x,y)^\beta dy \leq (4\pi t)^{-\frac{\beta}{2}} \int_{\mathbb{R}} e^{-\frac{\beta|x|^2}{4t}} dx = C_\beta t^{\frac{1-\beta}{2}}. \tag{3.23}$$

The solution in the the particular case $u_0 = 0$, $b = 0$, $\sigma = 1$ is

$$u(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y) W(ds,dy).$$

This stochastic integral exists and it is a Gaussian centered random variable with variance

$$\int_0^t \int_0^1 G_{t-s}(x,y)^2 dy ds = \int_0^t G_{2s}(x,x) ds < \infty,$$

because $G_{2s}(x,x) \leq C s^{-1/2}$. Notice that in dimension $d \geq 2$, $G_{2s}(x,x) \sim C s^{-1}$ and the variance is infinite. For this reason, the study of space-time white noise driven parabolic equations is restricted to the one-dimensional case.

In the case $\sigma \neq 0$ the stochastic integral is a Itô integral, defined using the isometry property

$$E\left(\left|\int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(ds,dy)\right|^2\right) = E\left(\int_0^t \int_0^1 G_{t-s}(x,y)^2 \sigma(u(s,y))^2 dy ds\right). \tag{3.24}$$

The following result asserts that Equation (3.20) has a unique solution if the coefficients are Lipschitz continuous.

Theorem 3.1 *Suppose that the coefficients b and σ are globally Lipschitz functions. Then there is a unique adapted process $u = \{u(t,x), t \geq 0, x \in [0,1]\}$ such that for all $T > 0$*

$$E\left(\int_0^T \int_0^1 u(t,x)^2 dxdt\right) < \infty,$$

and satisfies (3.20). Moreover, the solution u satisfies

$$\sup_{(t,x)\in[0,T]\times[0,1]} E(|u(t,x)|^p) < \infty \tag{3.25}$$

for all $p \geq 2$ and $T > 0$.

Proof. Consider the Picard iteration scheme defined by

$$u_0(t,x) = \int_0^1 G_t(x,y)u_0(y)dy$$

and

$$\begin{aligned} u_{n+1}(t,x) &= u_0(t,x) + \int_0^t \int_0^1 G_{t-s}(x,y)b(u_n(s,y))dyds \\ &\quad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u_n(s,y))W(dy,ds), \end{aligned} \tag{3.26}$$

$n \geq 0$. Using the Lipschitz condition on b and σ and the isometry property of the stochastic integral with respect to the two-parameter Wiener process (3.24), we obtain

$$\begin{aligned} &E(|u_{n+1}(t,x) - u_n(t,x)|^2) \\ &\quad \leq 2(T+1)\int_0^t \int_0^1 G_{t-s}(x,y)^2 E\left(|u_n(s,y) - u_{n-1}(s,y)|^2\right) dyds. \end{aligned}$$

Now we apply (3.23) with $\beta = 2$, and we obtain

$$\begin{aligned} &\int_0^1 E(|u_{n+1}(t,x) - u_n(t,x)|^2)dx \\ &\quad \leq C_T \int_0^t \int_0^1 E(|u_n(s,y) - u_{n-1}(s,y)|^2)(t-s)^{-\frac{1}{2}} dyds. \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_0^1 E(|u_{n+1}(t,x) - u_n(t,x)|^2)dx \\
\leq \; & C_T^2 \int_0^t \int_0^s \int_0^1 E(|u_n(r,z) - u_{n-1}(r,z)|^2)(s-r)^{-\frac{1}{2}}(t-s)^{-\frac{1}{2}} dz dr ds \\
= \; & C_T' \int_0^t \int_0^1 E(|u_n(r,z) - u_{n-1}(r,z)|^2) dz dr.
\end{aligned}$$

Iterating this inequality yields

$$\sum_{n=0}^{\infty} \sup_{t\in[0,T]} \int_0^1 E(|u_{n+1}(t,x) - u_n(t,x)|^2)dx < \infty.$$

This implies that the sequence $u_n(t,x)$ converges in $L^2([0,1]\times\Omega)$, uniformly in $t \in [0,T]$, to a stochastic process $u(t,x)$. The process $u(t,x)$ is adapted and satisfies (3.20). Uniqueness is proved by the same argument.

Let us now show (3.25). Fix $p > 6$. Applying Burkholder's inequality for stochastic integrals with respect to the Brownian sheet and the boundedness of the function u_0 yields

$$\begin{aligned}
E\left(|u_{n+1}(t,x)|^p\right) \leq \; & c_p \Big(\|u_0\|_\infty^p \\
& + E\left(\left(\int_0^t \int_0^1 G_{t-s}(x,y) \left| b(u_n(s,y)) \right| dyds \right)^p \right) \\
& + E\left(\left(\int_0^t \int_0^1 G_{t-s}(x,y)^2 \sigma(u_n(s,y))^2 dyds \right)^{\frac{p}{2}} \right) \Big) .
\end{aligned}$$

Using the linear growth condition of b and σ we can write

$$E\left(|u_{n+1}(t,x)|^p\right) \leq C_{p,T} \left(1 + E\left(\left(\int_0^t \int_0^1 G_{t-s}(x,y)^2 u_n(s,y)^2 dyds \right)^{\frac{p}{2}} \right) \right) .$$

Now we apply Hölder's inequality and (3.23) with $\beta = \frac{2p}{p-2} < 3$, and we obtain

$$\begin{aligned} E\left(|u_{n+1}(t,x)|^p\right) &\leq C_{p,T}\left(1 + \left(\int_0^t \int_0^1 G_{t-s}(x,y)^{\frac{2p}{p-2}} dyds\right)^{\frac{p-2}{2}} \right. \\ &\qquad \left. \times \int_0^t \int_0^1 E(|u_n(s,y)|^p) dyds \right) \\ &\leq C'_{p,T}\left(1 + \int_0^t \int_0^1 E(|u_n(s,y)|^p) dyds\right), \end{aligned}$$

and we conclude using Gronwall's lemma. ■

The next proposition tells us that the trajectories of the solution to the Equation (3.20) are α-Hölder continuous for any $\alpha < \frac{1}{4}$. For its proof we need the following technical inequalities.

(a) Let $\beta \in (1,3)$. For any $x \in [0,1]$ and $t, h \in [0,T]$ we have

$$\int_0^t \int_0^1 |G_{s+h}(x,y) - G_s(x,y)|^\beta dyds \leq C_{T,\beta} h^{\frac{3-\beta}{2}}, \qquad (3.27)$$

(b) Let $\beta \in (\frac{3}{2}, 3)$. For any $x, y \in [0,1]$ and $t \in [0,T]$ we have

$$\int_0^t \int_0^1 |G_s(x,z) - G_s(y,z)|^\beta dzds \leq C_{T,\beta} |x-y|^{3-\beta}. \qquad (3.28)$$

Proposition 3.2 *Let u_0 be a Hölder continuous function of order $\frac{1}{2}$ such that $u_0(0) = u_0(1) = 0$. Then, the solution u to Equation (3.20) satisfies*

$$E(|u(t,x) - u(s,y)|^p) \leq C_T(|t-s|^{\frac{p-4}{2}} + |x-y|^{\frac{p-2}{2}})$$

for all $s, t \in [0,T]$, $x, y \in [0,1]$, $p \geq 2$.

As a consequence, for any $\epsilon > 0$ the trajectories of the process $u(t,x)$ are Hölder continuous of order $1/4 - \epsilon$ in the variable t and Hölder continuous of order $1/2 - \epsilon$ in the variable x.

Proof. We will discuss only the term

$$U(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)\sigma(u(s,y))W(dy,ds).$$

Applying Burkholder's and Hölder's inequalities, we have for any $p > 6$

$$\begin{aligned}
&E(|U(t,x) - U(t,y)|^p) \\
&\quad \le C_p E\left(\left|\int_0^t \int_0^1 |G_{t-s}(x,z) - G_{t-s}(y,z)|^2 |\sigma(u(s,z))|^2 dzds\right|^{\frac{p}{2}}\right) \\
&\quad \le C_{p,T}\left(\int_0^t \int_0^1 |G_{t-s}(x,z) - G_{t-s}(y,z)|^{\frac{2p}{p-2}} dzds\right)^{\frac{p-2}{2}},
\end{aligned}$$

because $\int_0^T \int_0^1 E(|\sigma(u(s,z))|^p)dzds < \infty$. From (3.28) with $\beta = \frac{2p}{p-2}$, we know that this is bounded by $C|x-y|^{\frac{p-6}{2}}$.

On the other hand, for $t > s$ we can write

$$\begin{aligned}
&E(|U(t,x) - U(s,x)|^p) \\
&\le \; C_p \left\{ E\left(\left|\int_0^s \int_0^1 |G_{t-\theta}(x,y) - G_{s-\theta}(x,y)|^2 |\sigma(u(\theta,y))|^2 dyd\theta\right|^{\frac{p}{2}}\right)\right. \\
&\quad \left. + \, E\left(\left|\int_s^t \int_0^1 |G_{t-\theta}(x,y)|^2 |\sigma(u(\theta,y))|^2 dyd\theta\right|^{\frac{p}{2}}\right)\right\} \\
&\le \; C_{p,T}\left\{\left|\int_0^s \int_0^1 |G_{t-\theta}(x,y) - G_{s-\theta}(x,y)|^{\frac{2p}{p-2}} dyd\theta\right|^{\frac{p-2}{2}}\right. \\
&\quad \left. + \left|\int_0^s \int_0^1 G_{t-\theta}(x,y)^{\frac{2p}{p-2}} dyd\theta\right|^{\frac{p-2}{2}}\right\}.
\end{aligned}$$

Using (3.27) we can bound the first summand by $C_p|t-s|^{\frac{p-6}{4}}$. From (3.23) the second summand is bounded by

$$\begin{aligned}
\int_0^{t-s}\int_0^1 G_\theta(x,y)^{\frac{2p}{p-2}} dyd\theta \;&\le\; C_p \int_0^{t-s} \theta^{-\frac{p+2}{2(p-2)}} d\theta \\
&=\; C_p'|t-s|^{\frac{p-6}{2(p-2)}}.
\end{aligned}$$

As a consequence,

$$E(|U(t,x)-U(s,y)|^p) \le C_{p,T}\left(|x-y|^{\frac{p-6}{2}} + |t-s|^{\frac{p-6}{4}}\right),$$

■

3.1 Regularity of the probability law of the solution

In order to apply the criterion for absolute continuity, we will first show that the random variable $u(t,x)$ belongs to the space $\mathbb{D}^{1,2}$.

Proposition 3.3 *Let b and σ be C^1 functions with bounded derivatives. Then $u(t,x) \in \mathbb{D}^{1,2}$, and the derivative $D_{s,y}u(t,x)$ satisfies*

$$\begin{aligned} D_{s,y}u(t,x) &= G_{t-s}(x,y)\sigma(u(s,y)) \\ &\quad + \int_s^t \int_0^1 G_{t-\theta}(x,\eta) b'(u(\theta,\eta)) D_{s,y}u(\theta,\eta) d\eta d\theta \\ &\quad + \int_s^t \int_0^1 G_{t-\theta}(x,\eta)\sigma'(u(\theta,\eta)) D_{s,y}u(\theta,\eta) W(d\theta,d\eta) \end{aligned}$$

if $s < t$, and $D_{s,y}u(t,x) = 0$ if $s > t$.

That is, $\{D_{s,y}u(t,x), t \ge \theta\}$ is the solution of the stochastic heat equation

$$\frac{\partial D_{s,y}u}{\partial t} = \frac{\partial^2 D_{s,y}u}{\partial x^2} + b'(u)D_{s,y}u + \sigma'(u)D_{s,y}u\frac{\partial^2 W}{\partial t \partial x}$$

on $[s,\infty)\times[0,1]$, with Dirichlet boundary conditions and initial condition $\sigma(u(s,y))\delta_0(x-y)$.

On the other hand, Proposition 3.3 also holds if the coefficients are Lipschitz continuous. In this case, we replace $b'(u(t,x))$ and $\sigma'(u(t,x))$ by some bounded and adapted processes.

Proof. Consider the Picard approximations $u_n(t,x)$ introduced in (3.26). Suppose that $u_n(t,x) \in \mathbb{D}^{1,2}$ for all $(t,x) \in [0,T]\times[0,1]$ and

$$\sup_{(t,x)\in[0,T]\times[0,1]} E\left(\int_0^t \int_0^1 |D_{s,y}u_n(t,x)|^2 dyds\right) < \infty. \tag{3.29}$$

Applying the operator D to Eq. (3.26), we obtain that $u_{n+1}(t,x) \in \mathbb{D}^{1,2}$ and that

$$\begin{aligned} D_{s,y}u_{n+1}(t,x) &= G_{t-s}(x,y)\sigma(u_n(s,y)) \\ &\quad + \int_s^t \int_0^1 G_{t-\theta}(x,\eta)b'(u_n(\theta,\eta))D_{s,y}u_n(\theta,\eta)d\eta d\theta \\ &\quad + \int_s^t \int_0^1 G_{t-\theta}(x,\eta)\sigma'(u_n(\theta,\eta))D_{s,y}u_n(\theta,\eta)W(d\theta,d\eta). \end{aligned}$$

Note that

$$\begin{aligned} &E\left(\int_0^T \int_0^1 G_{t-s}(x,y)^2\sigma(u_n(s,y))^2 dyds\right) \\ &\qquad \leq C_1\left(1 + \sup_{t\in[0,T],x\in[0,1]} E(u_n(t,x)^2)\right) \leq C_2, \end{aligned}$$

for some constants $C_1, C_2 > 0$. Hence

$$\begin{aligned} &E\left(\int_0^t \int_0^1 |D_{s,y}u_{n+1}(t,x)|^2 dyds\right) \\ &\leq C_3\left(1 + E\left(\int_0^t \int_0^1 \int_s^t \int_0^1 G_{t-\theta}(x,\eta)^2|D_{s,y}u_n(\theta,\eta)|^2 d\eta d\theta dyds\right)\right) \\ &\leq C_4\left(1 + \int_0^t \sup_{\eta\in[0,1]} \int_s^t \int_0^1 (t-\theta)^{-\frac{1}{2}} E(|D_{s,y}u_n(\theta,\eta)|^2) d\theta dyds\right). \end{aligned}$$

Let

$$V_n(t) = \sup_{x\in[0,1]} E\left(\int_0^t \int_0^1 |D_{s,y}u_n(t,x)|^2 dyds\right).$$

Then

$$\begin{aligned} V_{n+1}(t) &\leq C_4\left(1 + \int_0^t V_n(\theta)(t-\theta)^{-\frac{1}{2}} d\theta\right) \\ &\leq C_5\left(1 + \int_0^t \int_0^\theta V_{n-1}(u)(t-\theta)^{-\frac{1}{2}}(\theta-u)^{-\frac{1}{2}} du d\theta\right) \\ &\leq C_6\left(1 + \int_0^t V_{n-1}(u)du\right) < \infty, \end{aligned}$$

due to (3.29). By iteration this implies that

$$\sup_{t\in[0,T],x\in[0,1]} V_n(t) < C,$$

where the constant C does not depend on n. Taking into account that $u_n(t,x)$ converges to $u(t,x)$ in $L^p(\Omega)$ for all $p \geq 1$, we deduce that $u(t,x) \in \mathbb{D}^{1,2}$, and $Du_n(t,x)$ converges to $Du(t,x)$ in the weak topology of $L^2(\Omega; H)$ (see Lemma 1.7). Finally, applying the operator D to both members of Eq. (3.20), we deduce the desired result. ∎

Furthermore, if the coefficients b and σ are are infinitely differentiable with bounded derivatives of all orders, then $u(t,x)$ belongs to $\mathbb{D}^\infty$.

Set $\gamma_{u(t,x)} = \int_0^t \int_0^1 (D_{s,y}u(t,x))^2 \, dyds$ and

$$B_{t,x} = \int_0^t \int_0^1 \sigma^2(u(s,y)) G_{t-s}^2(x,y) dsdy.$$

The following theorem asserts the existence and regularity of the density under strong ellipticity conditions.

Theorem 3.4 *Assume that the coefficients b and σ are Lipschitz functions, and $|\sigma(x)| \geq c > 0$ for all x. Then the law of $u(t,x)$ is absolutely continuous for all $t > 0$ and all $x \in (0,1)$. On the other hand, if b and σ are infinitely differentiable functions with bounded derivatives, then $u(t,x)$ has a C^∞ density for all (t,x) such that $t > 0$ and $x \in (0,1)$.*

Proof. Set $H = L^2([0,1])$. We have

$$\int_0^t \|D_s u(t,x)\|_H^2 ds \geq \frac{1}{2} \int_{t-\delta}^t \|G_{t-s}(x-\cdot)\sigma(u(s,\cdot))\|_H^2 ds - I_\delta, \tag{3.30}$$

where

$$\begin{aligned} I_\delta &= \int_{t-\delta}^t \| \int_s^t \int_0^1 G_{t-r}(x-z)\sigma'(u(r,z))D_s u(r,z) W(dr,dz) \\ &\quad + \int_s^t \int_0^1 G_{t-r}(x-z) b'(u(r,z)) D_s u(r,z) dr \|_H^2 ds. \end{aligned}$$

The first term in the inequality (3.30) can be bounded below by a constant times δ, while the term I_δ is of order δ^γ for some $\gamma > 0$. Using these ideas it is not difficult to show that $E(gamma_{u(t,x)}^{-p} < \infty$ for all $p \geq 2$. ∎

• In [10] Pardoux and Zhang proved that $u(t,x)$ has an absolutely continuous distribution for all (t,x) such that $t>0$ and $x\in(0,1)$, if the coefficients b and σ are Lipschitz continuous and $\sigma(u_0(y))\neq 0$ for some $y\in(0,1)$.

• Bally and Pardoux considered in [2] the Equation (3.20) with Neumann boundary conditions on $[0,1]$, assuming that the coefficients b and σ are infinitely differentiable functions, which are bounded together with their derivatives. The main result of this paper says that the law of any vector of the form $(u(t,x_1),\dots,u(t,x_d))$, $0\le x_1\le\cdots\le x_d\le 1$, $t>0$, has a smooth and strictly positive density with respect to the Legesgue measure on the set $\{\sigma>0\}^d$.

4 Spatially homogeneous SPDEs

We are interested in the following general class of stochastic partial differential equations

$$Lu(t,x)=\sigma(u(t,x))\dot{W}(t,x)+b(u(t,x)), \tag{4.31}$$

$t\ge 0$, $x\in\mathbb{R}^d$, where L denotes a second order differential operator, and we impose the initial conditions

$$u(0,x)=\frac{\partial u}{\partial t}(0,x)=0.$$

Assumptions:

(H1) The fundamental solutions to $Lu=0$ denoted by Γ is a non-negative measure of the form $\Gamma(t,dy)dt$ such that $\Gamma(t,\mathbb{R}^d)\le C_T<\infty$ for all $0\le t\le T$ and all $T>0$.

(H2) The noise W is a zero mean Gaussian family $W=\{W(\varphi),\varphi\in C_0^\infty(\mathbb{R}^{d+1})\}$ with covariance

$$E(W(\varphi)W(\psi))=\int_0^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(t,y)f(x-y)\psi(t,y)dxdydt, \tag{4.32}$$

where f is a non-negative continuous function of $\mathbb{R}^d\backslash\{0\}$ such that it is the Fourier transform of a non-negative definite tempered measure μ on $\mathbb{R}^d$.

That is,

$$f(x)=\int_{\mathbb{R}^d}\exp(-2i\pi x\cdot\xi)\mu(d\xi),$$

and there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-m} \mu(d\xi) < \infty.$$

Then, the covariance (4.32) can also be written, using Fourier transform, as

$$E(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}\varphi(s)(\xi)\overline{\mathcal{F}\psi(s)(\xi)}\mu(d\xi)ds.$$

The completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing C^∞ functions, endowed with the inner producrt

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) f(x-y)\psi(y)dxdy$$

is denoted by $\mathcal{H}$. Notice that $\mathcal{H}$ may contain distributions. Set $\mathcal{H}_T = L^2([0,T];\mathcal{H})$.

By definition, the solution to (4.31) is an adapted stochastic process $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$, satisfying

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(t,x))W(ds,dy) + \int_0^t \int_{\mathbb{R}^d} b(u(t-s,x-y))\Gamma(s,dy). \tag{4.33}$$

The stochastic integral appearing in formula (4.33) requires some care because the integrand is a measure. We refer to Dalang [4] for the construction of this integral. The main ideas are as follows.

4.1 Stochastic integrals

Fix a time interval $[0,T]$. The integral of an elementary process of the form

$$g(s,x) = \mathbf{1}_{(a,b]}(s)\mathbf{1}_A(x)X,$$

where $0 \leq a < b \leq T$, $A \in \mathcal{B}(\mathbb{R}^d)$ and X is a bounded an $\mathcal{F}_a$-measurable random variable is defined as

$$g \cdot W = W((a,b] \times A)X.$$

This definition is extended by linearity to the set $\mathcal{E}$ of all finite linear combinations of elementary processes. We have

$$E(|g \cdot W|^2) = E(X^2)(b-a) \int_{\mathbb{R}^d} \mathbf{1}_A(x) f(x-y) \mathbf{1}_A(y) dx dy = \|g\|^2_{L^2(\Omega;\mathcal{H}_T)}. \tag{4.34}$$

The σ-field on $\Omega \times \mathbb{R}_+$ generated by elements of the form $\mathbf{1}_{(a,b]}(s)X$, is called the *predictable* σ-field and denoted by $\mathcal{P}$. The completion of $\mathcal{E}$ with respect to the norm of $L^2(\Omega;\mathcal{H}_T)$ is equal to the class of square integrable $\mathcal{H}$-valued predictable processes:

$$\overline{\mathcal{E}} = L^2(\Omega \times [0,T], \mathcal{P}, P \times dt; \mathcal{H}).$$

The stochastic integral $g \cdot W$ can be extended to the space $L^2(\Omega;\mathcal{H}_T)$, and the isometry property (4.34) is preserved.

The following proposition provides a useful example of a random distribution which belongs to the space $L^2(\Omega;\mathcal{H}_T)$. We need the following condition on the fundamental solution $\Gamma(t,dx)$:

$$\int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt < \infty. \tag{4.35}$$

Proposition 4.1 *Suppose that* $\{Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ *is a predictable process, continuous in* $L^2(\Omega)$*, such that*

$$C_Z := \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(Z(t,x)^2) < \infty. \tag{4.36}$$

Then if (4.35) holds, $\Gamma(t,dx)Z(t,x)$ *belongs to* $L^2(\Omega;\mathcal{H}_T)$*.*

Proof. Fix $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi \geq 0$, the support of ψ is contained in the unit ball of $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} \psi(x)dx = 1$. For $n \geq 1$ set $\psi_n(x) = n^d \psi(nx)$ and $\Gamma_n(t) = \psi * \Gamma(t)$. Then for each t, $\Gamma_n(t) \in \mathcal{S}(\mathbb{R}^d)$. Then the sequence $\Gamma_n(t,x)Z(t,x)$ is bounded in $L^2(\Omega;\mathcal{H}_T)$.

In fact, we have

$$
\begin{aligned}
& E\int_0^T\int_{\mathbb{R}^d}|\mathcal{F}[\Gamma_n(t,x))Z(t,x)](\xi)|^2\mu(d\xi)dt \\
& = E\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_n(t,x)Z(t,x)f(x-y)\Gamma_n(t,y)Z(t,y)dt \\
& \leq C_Z\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_n(t,x)f(x-y)\Gamma_n(t,y)dt \\
& = C_Z\int_0^T\int_{\mathbb{R}^d}|\mathcal{F}\Gamma_n(t)(\xi)|^2\mu(d\xi)dt \\
& \leq C_Z\int_0^T\int_{\mathbb{R}^d}|\mathcal{F}\Gamma(t)(\xi)|^2\mu(d\xi)dt < \infty.
\end{aligned}
$$

On the other hand, the sequence $\Gamma_n(t,x)Z(t,x)$ converges weakly in $L^2(\Omega;\mathcal{H}_T)$ to $\Gamma(t,dx)Z(t,x)$. In fact, for any element $\varphi\in\mathcal{H}$, any bounded random variable Y and any $0\leq s\leq t$ we have

$$
\begin{aligned}
& E\left(Y\int_s^t\int_{\mathbb{R}^d}\mathcal{F}[\Gamma_n(r,x))Z(r,x)](\xi)\overline{\mathcal{F}\varphi(\xi)}\mu(d\xi)dr\right) \\
& = E\left(Y\int_s^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma_n(r,x)Z(r,x)f(x-y)\varphi(y)dxdy\right) \\
& = \int_s^t\int_{\mathbb{R}^d}\Gamma(r,dz)\int_{\mathbb{R}^d}\psi_n(w)E(YZ(r,z+w))(\int_{\mathbb{R}^d}\varphi(y)f(z+w-y)dy)dw,
\end{aligned}
$$

and this converges as n tends to infinity to

$$
\int_s^t\int_{\mathbb{R}^d}\Gamma(r,dz)E(YZ(r,z))\int_{\mathbb{R}^d}\varphi(y)f(z-y)dy
$$

because the functions $w\rightarrow E(YZ(r,z+w))$ and $w\rightarrow\int_{\mathbb{R}^d}\varphi(y)f(z+w-y)dy$ are continuous. ■

Under the assumptions of Proposition 4.1, suppose in addition that

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}E(|Z(t,x)|^p)<\infty,
$$

for some $p \geq 2$. Then

$$E\left(\left|\int_0^T \int_{\mathbb{R}^d} \Gamma(t,dx)Z(t,x)W(dt,dx)\right|^p\right) \leq c_p(\nu_t)^{\frac{p}{2}-1} \times \int_0^T \left(\sup_{x\in\mathbb{R}^d} E(|Z(t,x)|^p)\right) \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi)ds,$$

where

$$\nu_t = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi)ds.$$

4.2 Existence and uniqueness of solutions

The following theorem gives the existence and uniqueness of a solution for Equation (4.31) (see Dalang [4]).

Theorem 4.2 *Suppose that the fundamental solution of $Lu = 0$ satisfies Hypothesis (4.35) for all $T > 0$. Then the Equation (4.31) has a unique solution $u(t,x)$ which is continuous in L^2 and satisfies*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < \infty,$$

for all $T > 0$ and $p \geq 1$.

Example 1. *The wave equation.* Let Γ_d the fundamental solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0.$$

We know that

$$\Gamma_1(t) = \frac{1}{2}\mathbf{1}_{\{|x|<1\}},$$
$$\Gamma_2(t) = c_2(t^2 - |x|^2)_+^{-1/2},$$
$$\Gamma_3(t) = \frac{1}{4\pi}\sigma_t,$$

where σ_t denotes the surface measure on the 3-dimensional sphere of radius t. In particular, for each t, $\Gamma_i(t)$ has compact support. Furthermore, for all dimensions d

$$\mathcal{F}\Gamma_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Notice that only in dimensions $d = 1, 2, 3$, Γ_d is a measure. Elementary estimates show that there are positive constants c_1 and c_2 depending on T such that

$$\frac{c_1}{1+|\xi|^2} \leq \int_0^T \frac{\sin^2(2\pi t|\xi|)}{4\pi^2|\xi|^2} ds \leq \frac{c_2}{1+|\xi|^2}.$$

Therefore, Γ_d satisfies condition (4.35) if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{4.37}$$

Example 2. *The heat equation.* Let Γ be the fundamental solution to the heat equation

$$\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = 0.$$

Then,

$$\Gamma(t,x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$$

and

$$\mathcal{F}\Gamma(t)(\xi) = \exp(-4\pi^2 t|\xi|^2).$$

Because

$$\int_0^T \exp(-4\pi^2 t|\xi|^2) = \frac{1}{4\pi^2|\xi|^2}(1-\exp(-4\pi^2 T|\xi|^2)),$$

we conclude that condition (4.35) holds if an only if (4.37) holds.

Condition (4.37) can be expressed in terms of the covariance function f as follows:

For $d = 1$, (4.37) always holds.

For $d = 2$, (4.37) holds if and only if

$$\int_{|x|\leq 1} f(x) \log \frac{1}{|x|} dx < \infty.$$

For $d \geq 2$, (4.37) holds if and only if

$$\int_{|x|\leq 1} f(x) \frac{1}{|x|^{d-2}} dx < \infty.$$

4.3 Regularity of the law

Assume that the coefficients σ and b and C^1 functions with bounded derivatives. Then for any $(t,x) \in [0,T] \times \mathbb{R}^d$, the random variable $u(t,x)$ belongs to $\mathbb{D}^{1,p}$ for any $p \geq 2$. Moreover, the derivative $Du(t,x)$ is an $\mathcal{H}$-valued process which satisfies the following linear stochastic differential equation

$$\begin{aligned} D_s u(t,x) &= \Gamma(t-s, x-dy)\sigma(u(s,y)) \\ &\quad + \int_s^t \int_{\mathbb{R}^d} \Gamma(t-r, x-z)\sigma'(u(r,z)) D_s u(r,z) W(dr,dz) \\ &\quad + \int_s^t \int_{\mathbb{R}^d} \Gamma(t-r, dz) b'(u(r,x-z)) D_s u(r, x-z) dr. \end{aligned}$$

Theorem 4.3 *Suppose that $|\sigma(z)| \geq c > 0$ for all z. Then, for all $t > 0$ and $x \in \mathbb{R}^d$ the random variable $u(t,x)$ has an absolutely continuous distribution.*

Proof. It suffices to show that $\|Du(t,x)\|_{\mathcal{H}_T} > 0$ almost surely. We have

$$\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds \geq \frac{1}{2} \int_{t-\delta}^t \|\Gamma(t-s, x-dy)\sigma(u(s,y))\|_{\mathcal{H}}^2 ds - I_\delta,$$

where

$$\begin{aligned} I_\delta &= \int_{t-\delta}^t \| \int_s^t \int_{\mathbb{R}^d} \Gamma(t-r, x-z)\sigma'(u(r,z)) D_s u(r,z) W(dr,dz) \\ &\quad + \int_s^t \int_{\mathbb{R}^d} \Gamma(t-r, dz) b'(u(r,x-z)) D_s u(r,x-z) dr \|_{\mathcal{H}}^2 ds \\ &\quad \leq 2I_{\delta,1} + 2I_{\delta,2} \end{aligned}$$

We have

$$
\begin{aligned}
&\int_{t-\delta}^{t} \|\Gamma(t-s,x-dy)\sigma(u(s,y))\|_{\mathcal{H}}^2 ds = \int_0^{\delta} \|\Gamma(s,x-dy)\sigma(u(t-s,y))\|_{\mathcal{H}}^2 ds \\
&= \int_0^{\delta}\int_{\mathbb{R}^d} |\mathcal{F}[\Gamma(s,x-dy)\sigma(u(t-s,y))(\xi)]|^2 \mu(d\xi) ds \\
&= \int_0^{\delta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(s,x-dy)\Gamma(s,x-dz) f(y-z)\sigma(u(t-s,y))\sigma(u(t-s,z)) ds \\
&\geq c^2 \int_0^{\delta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(s,x-dy)\Gamma(s,x-dz) f(y-z) ds \\
&= c^2 \int_0^{\delta}\int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds := c^2 g(\delta).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
E(I_{\delta,1}) &= \int_0^{\delta} E\left(\| \int_0^s \int_{\mathbb{R}^d} \Gamma(r,x-z)\sigma'(u(t-r,z)) D_{t-s}u(t-r,z) W(dr,dz) \|_{\mathcal{H}}^2 \right) ds \\
&= E\int_0^{\delta}\int_0^s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,x-dz)\Gamma(r,x-dy) f(y-z) \\
&\quad \times \sigma'(u(t-r,z))\sigma'(u(t-r,y)) \langle D_{t-s}u(t-r,z), D_{t-s}u(t-r,y)\rangle_{\mathcal{H}} dydzdrds \\
&\leq \|\sigma'\|_{\infty}^2 \sup_{x\mathbb{R}^d} E\left(\int_{t-\delta}^{t} \|D_s u(t,x)\|_{\mathcal{H}}^2\right) ds) \\
&\quad \times \int_0^{\delta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,x-dz)\Gamma(r,x-dy) f(y-z) dydzdr \\
&= \|\sigma'\|_{\infty}^2 \sup_{x\mathbb{R}^d} E(\int_{t-\delta}^{t} \|D_s u(t,x)\|_{\mathcal{H}}^2) ds) g(\delta),
\end{aligned}
$$

and

$$
\begin{aligned}
E(I_{\delta,2}) &= \int_0^\delta E\left(\| \int_0^s \int_{\mathbb{R}^d} \Gamma(r,dz) b'(u(t-r,x-z)) D_{t-s}u(t-r,x-z)dr\|_{\mathcal{H}}^2\right) ds \\
&= E\int_0^\delta \int_0^s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,dz)\Gamma(r,dy) b'(u(t-r,x-z)) b'(u(t-r,x-y)) f(y-z) \\
&\quad \times \langle D_{t-s}u(t-r,x-z), D_{t-s}u(t-r,x-y)\rangle_{\mathcal{H}} dydzdrds \\
&\le \|b'\|_\infty^2 \sup_{x\mathbb{R}^d} E(\int_{t-\delta}^t \|D_s u(t,x)\|_{\mathcal{H}}^2) ds) \\
&\quad \times \int_0^\delta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Gamma(r,dz)\Gamma(r,dy) f(y-z) dydzdr \\
&= \|b'\|_\infty^2 \sup_{x\mathbb{R}^d} E(\int_{t-\delta}^t \|D_s u(t,x)\|_{\mathcal{H}}^2) ds) g(\delta).
\end{aligned}
$$

The stationary property of the random field $u(t,x)$ implies that $A_\delta := E(\int_{t-\delta}^t \|D_s u(t,x)\|_{\mathcal{H}}^2) ds)$ does not depend on x. Hence, we obtain, assuming $\frac{1}{n} \le \frac{c^2 g(\delta)}{2}$

$$
\begin{aligned}
& P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \frac{1}{n}\right) \le P\left(I_\delta \ge \frac{c^2}{2} g(\delta) - \frac{1}{n}\right) \\
\le \;& (\frac{c^2}{2} g(\delta) - \frac{1}{n})^{-1} E(I_\delta) \le (\frac{c^2}{2} g(\delta) - \frac{1}{n})^{-1} 2\left(\|b'\|_\infty + \|\sigma'\|_\infty^2\right) A_\delta g(\delta).
\end{aligned}
$$

Therefore

$$
\lim_n P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \frac{1}{n}\right) \le \frac{4}{c^2}\left(\|b'\|_\infty + \|\sigma'\|_\infty^2\right) A_\delta,
$$

which converges to zero as δ tends to zero. Hence,

$$
P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds = 0\right) = 0.
$$

■

If the coefficients b and σ are infinitely differentiable with bounded derivatives of all orders, then for all $t \ge 0$ and $x \in \mathbb{R}^d$ the random variable $u(t,x)$ belongs to the space $\mathbb{D}^\infty$. Then, we have the following result on the regularity of the density.

Theorem 4.4 *Suppose that there exists a constant $\gamma > 0$ such that for all $t > 0$,*

$$
\int_0^t \int_{\mathbb{R}^d} ds |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) \le Ct^\gamma. \tag{4.38}
$$

Assume $|\sigma(x)| \geq c > 0$. *Then, the law of* $u(t,x)$ *has a* C^∞ *density for all* $t > 0$ *and* $x \in \mathbb{R}^d$.

Proof. In order to show this result we need to prove that

$$E\left(\left|\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds\right|^{-p}\right) < \infty$$

for all $p \geq 2$. Proceeding as before we get

$$\begin{aligned} P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \epsilon\right) &\leq P\left(I_\delta \geq \frac{c^2}{2} g(\delta) - \epsilon\right) \\ &\leq (\frac{c^2}{2} g(\delta) - \epsilon)^{-p} E(I_\delta^p). \end{aligned}$$

As before we can get the following estimates

$$\begin{aligned} E(I_{\delta,1}^p) &\leq \delta^{p-1} \int_0^\delta E\left(\| \int_0^s \int_{\mathbb{R}^d} \Gamma(r, x-z)\sigma'(u(t-r,z)) D_{t-s} u(t-r,z) W(dr,dz)\|_{\mathcal{H}}^{2p}\right) ds \\ &\leq c_p \delta^{p-1} E \int_0^\delta \Bigg| \int_0^s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma(r, x-dz)\Gamma(r,x-dy) f(y-z) \\ &\quad \times \sigma'(u(t-r,z))\sigma'(u(t-r,y)) \langle D_{t-s}u(t-r,z), D_{t-s}u(t-r,y)\rangle_{\mathcal{H}} dydzdr \Bigg|^p ds \\ &\leq c_p \delta^{p-1} \|\sigma'\|_\infty^{2p} E((\int_{t-\delta}^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds)^p) g(\delta)^p, \end{aligned}$$

and similarly,

$$E(I_{\delta,2}^p) \leq c_p \delta^{p-1} \|b'\|_\infty^{2p} E((\int_{t-\delta}^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds)^p) g(\delta)^p.$$

Set $B_p(\delta) = E((\int_{t-\delta}^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds)^p)$. Then,

$$P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \epsilon\right) \leq (\frac{c^2}{2} g(\delta) - \epsilon)^{-p} c_p(\|b'\|_\infty^{2p} + \|b'\|_\infty^{2p}) \delta^{p-1} B_p(\delta) g(\delta)^p.$$

Suppose that we choose δ in such a way that $g(\delta) = \frac{1}{c^2}\epsilon$. Then,

$$P\left(\int_0^t \|D_s u(t,x)\|_{\mathcal{H}}^2 ds < \epsilon\right) \leq C_p \delta^{p-1} B_p(\delta).$$

To finish the argument, we need that $\delta \le C\epsilon^\gamma$ for some $\gamma > 0$, and this is possible by condition (4.38). ■

Remark 1 Condition (4.38) is satisfied in the case of the wave equation in dimension 1,2,3 and for the heat equation in dimension 1 if (4.37) holds.

Remark 2 Theorems 4.3 and 4.4 generalize those proved in the references [11] and [12].

References

[1] N. Bouleau and F. Hirsch: *Dirichlet Forms and Analysis on Wiener Space*. de Gruyter Studies in Math. 14, Walter de Gruyter, 1991.

[2] V. Bally and E. Pardoux: Malliavin calculus for white noise driven Parabolic SPDEs. *Potential Analysis* **9**(1998) 27–64.

[3] R. Cairoli and J. B. Walsh: Stochastic integrals in the plane. *Acta Mathematica* **134** (1975) 111–183.

[4] R. Dalang: Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* **4** (1999), no. 6, 29 pp.

[5] K. Itô: Multiple Wiener integral. *J. Math. Soc. Japan* **3** (1951) 157–169.

[6] A. Millet, M. Sanz-Solé: A stochastic wave equation in two space dimension: Smoothness of the law. *Ann. Probab.* **27** (1999) 803-844.

[7] C. Mueller: On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.* **37** (1991) 225–245.

[8] D. Nualart: *The Malliavin Calculus and related topics.* 2nd edition. Springer-Verlag 2006.

[9] E. Pardoux and T. Zhang: Absolute continuity of the law of the solution of a parabolic SPDE. *J. Functional Anal.* **112** (1993) 447–458.

[10] Ll. Quer-Sardanyons and M. Sanz-Solé: Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation. *J. Funct. Anal.* **206** (2004) 1–32.

[11] Ll. Quer-Sardanyons and M. Sanz-Solé: A stochastic wave equation in dimension 3: smoothness of the law. *Bernoulli* **10** (2004) 165–186.

[12] J. B. Walsh: An introduction to stochastic partial differential equations. In: *Ecole d'Ete de Probabilites de Saint Flour XIV,* Lecture Notes in Mathematics **1180** (1986) 265-438.