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Viscosity Solutions to PDEs: an introduction.

Outlines

- Introduction to viscosity theory $\left\langle \begin{array}{l} \text{examples.} \\ \text{definitions.} \end{array} \right.$

- The p-Laplacian & ∞-Laplacian.

- Comparison principle and Perron's method.

- Boundary value problems.

References

1. User's guide to viscosity solutions of second order PDEs
Crandall, Ishii & Lions, Bull of AMS, 1992.
2. PDEs, Evans, 1997.

Consider Possion's equation $\begin{cases} -\Delta u = f, & \Omega \\ u = 0, & \partial\Omega \end{cases}$ (1)

$\Omega \subset \mathbb{R}^N$ is a smooth domain in $\mathbb{R}^N$.

If f is smooth and Ω has simple geometry, then we obtain a classical solution using Green's funtion:

$$u(x) = \int_\Omega f(y) G(x,y)\,dy .$$

If f is not smooth enough, say $f \in L^2(\Omega)$, then one can only obtain a solution in the weak/generalized sense.

$\exists\ u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v \quad , \quad \forall v \in W_0^{1,2}(\Omega).$$

Regularity theory asserts that weak solutions are classical solutions if the data function f is smooth.

The theory of viscosity solution applies to certain PDEs which allow continuous functions to be solutions.

Consider $F(x, u, Du, D^2u) = 0$

where $F: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$

and $S(N)$ = set of symmetric $N \times N$ matrices

- F is monotone in r if
$$F(x, r, p, X) \leq F(x, s, p, X) \text{ whenever } r \leq s. \qquad (2)$$
- F is degenerate elliptic if
$$F(x, r, p, X) \leq F(x, r, p, Y) \text{ whenever } X \geq Y. \qquad (3)$$
- F is proper if F satisfies both (2) & (3).

Note that the inequality $X \geq Y$ means
$$\langle Xq, q\rangle \geq \langle Yq, q\rangle \quad \forall q \in \mathbb{R}^N.$$

Examples:

① Laplace's equations $\quad -\Delta u + c(x)u = f(x)$.

Then $F(x,r,p,X) = -\operatorname{trace}(X) + c(x)r - f(x)$.

F is proper if $c \geq 0$.

② Degenerate elliptic linear equations

$$-\sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x).$$

where $A = [a_{ij}]$ is symmetric. Then

$$F(x,r,p,X) = -\operatorname{trace}(A(x)X) + \sum b_i(x)p_i + c(x)r - f(x).$$

F is
- degenerate elliptic if $A(x) \geq 0$.
- proper if $c(x) \geq 0$ & $A(x) \geq 0$.
- uniformly elliptic if $\lambda I \leq A(x) \leq \Lambda I$.

③ Quasilinear elliptic equations

$$-\sum a_{i,j}(x, Du)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x,u,Du) = 0,$$

where $A(x,p) = [a_{ij}(x,p)] \in S(N)$. Then

$$F(x,r,p,X) = -\operatorname{trace}(A(x,p)X) + b(x,r,p).$$

For example, $-\mu\Delta u + f(x,u,Du) = 0$, with $\mu > 0$,
is regarded as first order Hamilton-Jacobi equation
perturbed by an additional "viscosity term" $-\mu\Delta u$.
Equations of this type arise in optimal stochastic control.

④ Parabolic problems $\quad u_t + F(t,x,u,Du,D^2u) = 0$,

when considered as an equation in the $(N+1)$ independent variables (t,x), is proper if $(x,r,p,X) \longrightarrow F(t,x,r,p,X)$ is proper for each fixed $t \in [0,T]$.

THE NOTIONS OF VISCOSITY SOLUTIONS

Let $u \in C^2(\mathbb{R}^N)$ be such that

$$F(x, u, Du, D^2u) \leqslant 0.$$

Suppose φ is C^2 and $\hat{x}$ is a local max of $(u-\varphi)$.
Then $Du(\hat{x}) = D\varphi(\hat{x})$ and $D^2u(\hat{x}) \leqslant D^2\varphi(\hat{x})$; so, by degenerate ellipticity,

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leqslant 0.$$

We note that the inequality does not depend on the derivatives of u.

Definition 1

Let F be degenerate elliptic and continuous, $\Omega \subset \mathbb{R}^N$.

i) A viscosity subsolution of $F=0$ (equivalently, a viscosity solution of $F \leqslant 0$) on Ω is a function $u \in USC(\Omega)$ (upper semi-continuous $\Omega \to \mathbb{R}$) such that

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leqslant 0, \tag{4}$$

whenever φ is C^2 and $x \in \Omega$ is a local maximum of $u-\varphi$.

ii) Similarly, a viscosity supersolution of $F=0$ on Ω is a function $u \in LSC(\Omega)$ (lower semi-continuous $\Omega \to \mathbb{R}$) such that

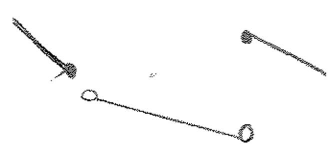
$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \geqslant 0 \tag{5}$$

whenever φ is C^2 and $x \in \Omega$ is a local minimum of $u-\varphi$.

iii) Finally, u is a viscosity solution of $F=0$ if it is both a viscosity subsolution & a viscosity supersolution.

Remark 2

The requirement that a subsolution is upper semi-cont is to produce maxima of $(u-\varphi)$.



Recall $u \in USC(\Omega)$

$\Leftrightarrow$ $\{x : u(x) \geqslant c\}$ is closed in Ω.

Remark 3

In definition 1, extrema of $(u-\varphi)$ can be strict. Precisely, $u \in USC(\Omega)$ is a vis. subsolution if

$$F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) \leq 0$$

whenever φ is C^2, $\varphi(x) = u(x)$ and $u(y) < \varphi(y)$ for $y \neq x$.

The definition of vis. supersolution follows in a similar way.

Remark 4

It is straightforward that if u is a classical sub/super solution then u is a viscosity sub/super-solution.