An End to End Gluing Construction for Metrics of Constant Positive Scalar Curvature

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1   Introduction

The goal of this paper to to describe a general process by which one can glue together metrics of constant positive scalar curvature on punctured spheres along their ends to obtain new metrics of constant positive scalar curvature.

First let $(M_1 = S^n \backslash \{p_0 \ldots p_{k_1-1}\}, g_1)$ and $(M_2 = S^n \backslash \{q_0 \ldots q_{k_2-1}\}, g_2)$ be complete metrics of scalar curvature $n(n-1)$. These metrics are asymptotic to Delaunay metrics in small (standard spherical) punctured balls about p_j and q_j respectively. We will refer to these punctured balls as the ends of M_1 and M_2. The Delaunay metrics can be written as

$$u_\epsilon^{\frac{4}{n-2}}(t+T)(dt^2 + d\theta^2)$$

where u_ϵ is a periodic function which assumes its minimal value ϵ (called the necksize of the metric) at $t = 0$. These metrics are uniquely determined by their singular set on S^n, necksize and the translation parameter T. Assume that we choose g_1 and g_2 such that the asymptotic necksize of g_1 at p_0 is equal to the asymptotic necksize of g_2 at q_0; we will call this common necksize ϵ. Then we can truncate M_1 and M_2 by removing small spherical balls around p_0 and q_0 and patch these two metrics together at a neck (value of t where u_ϵ achieves its minimum) to obtain a new metric $\tilde{g}$ on $M = S^n \backslash \{p_1 \ldots p_{k_1-1}, q_1 \ldots q_{k_2-1}\}$. This construction depends on two parameters: R, which we can think of as the size of the balls we excised in the truncation process, and $\phi \in SO(n)$ which specifies a rotation in the S^{n-1} factor of the second summand. Notice that the parameter R is discrete. To indicate the dependence of $\tilde{g}$ on these parameters we will denote it as $\tilde{g}_{R,\phi}$. Much of the analysis is independent of at least one of these parameters, and in this case we will suppress the appropriate subscript. We will construct this metric, which we will call an approximate solution (because its scalar curvature is very close to $n(n-1)$), in section 4.

The metric $\tilde{g}_{R,\phi}$ does not have constant scalar curvature, but the deviation $\psi = n(n-1) - \tilde{S}_{R,\phi}$ is globally small. More precisely, without any modification to g_1 and g_2, ψ is compactly supported and $\|\psi\|_{C^{0,\alpha}(M)} = O(e^{-R})$. After modifying g_1 and g_2 by conformal transformations, we can further arrange that $\|\psi\|_{C^{0,\alpha}(M)} = O(e^{-\gamma_{n+1}(\epsilon)R})$ where $\gamma_{n+1}(\epsilon)$ is a coefficient we will discuss in section 3. We wish to deform $\tilde{g}_{R,\phi}$ by a conformal factor to obtain a metric with scalar curvature $n(n-1)$. Recall how the scalar curvature transforms under a conformal change of metric: if $g' = u^{\frac{4}{n-2}} g$ then

$$S_{g'} = S_g u^{-\frac{4}{n-2}} - \frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \Delta_g u,$$

which we can rewrite as

$$\Delta_g u - \frac{n-2}{4(n-1)} S_g u + \frac{n-2}{4(n-1)} S_{g'} u^{\frac{n+2}{n-2}} = 0. \tag{1}$$

If we normalize the scalar curvatures by setting $S_g = n(n-1) - \psi$ and $S_{g'} = n(n-1)$ the above equation becomes

$$\Delta_g u - \frac{n(n-2)}{4}u + \left(\frac{n-2}{n-1}\right)(\psi/4)u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0.$$

The linearized equation (linearized about $u = 1$) is

$$L_g(u) = \Delta_g u + nu + \left(\frac{n-2}{n-1}\right)(\psi/4)u = 0. \tag{2}$$

We will call L_g the Jacobi operator associated to g and solutions $L_g u = 0$ Jacobi fields of g. Notice that $L_g = \Delta_g + n$ if $\psi = 0$, i.e. if we wish to deform a constant scalar curvature metric into another conformal constant scalar curvature metric. In the present case $L_{\tilde{g}_{R,\phi}}$ is a small perturbation of $\Delta_{\tilde{g}_{R,\phi}} + n$.

Our first step is to show that we can solve this linearized equation, with uniform (in R) estimates on the size of the solution operator. One can also think of this step as finding a positive lower bound on the spectrum of $L_{\tilde{g}_{R,\phi}}$ as an operator between appropriate function spaces. We will address precisely which function spaces are the proper ones for this problem in section 5.1. In order to prove this we will need to assume that (M_i, g_i) are both unmarked nondegenerate; i.e. there are no Jacobi fields which decay at a rate faster than e^{-t_j} on all ends. We will also need to assume that one can adjust the necksize of the end corresponding to p_0 in the moduli space of constant scalar curvature metrics (see the statement of the theorem for a precise statement of this condition). We need this last condition to exclude certain Jacobi fields which we could glue together to yield an exponentially small eigenvalue. This linear analysis will occupy section 5.

Then in section 6.1 we will explicitly write down geometric deformations of the metric associated to a parameter $u \in \mathbb{R}^{k(2n+2)}$, where $k = k_1 + k_2 - 2$ is the number of ends of M. One can think of these deformations as adjusting the necksizes and position of the necks of the metric on the ends of M. Finally, in section 6.2 we will use the solution operator we found and these geometric deformations to solve the nonlinear problem via the Contraction Mapping principle. This will yield the existence part of the following theorem.

Theorem 1 *Let $(M_1 = S^n \backslash \{p_0 \ldots p_{k_1-1}\}, g_1)$ and $(M_2 = S^n \backslash \{q_0, \ldots q_{k_2-1}\}, g_2)$ be complete metrics with scalar curvature $n(n-1)$. Assume g_1 and g_2 are unmarked nondegenerate and that the asymptotic necksizes associated to p_0 and q_0 are both ϵ. Assume also that there is a one-parameter family of scalar curvature $n(n-1)$ metrics g_t on M_1, $t \in (-\delta, \delta)$, where the asymptotic necksize of g_t associated to p_0 is $\epsilon + t$. Then for $\eta > 0$ there is an R_0 such that for $R \geq R_0$ one can deform the approximate solution $(M_R, \tilde{g}_R)$ first by a geometric parameter u with $|u| < \eta$ and then by a conformal factor $(1 + v)^{\frac{4}{n-2}}$ (with v exponentially decaying) to obtain a metric with scalar curvature $n(n-1)$. Moreover, this metric is unmarked nondegenerate.*

We first remark that the connect sum of two Delaunay metrics constructed in [MPU1] and all the metrics constructed by Byde in [B] and by Mazzeo and Pacard in [MP] satisfy all the hypotheses of this theorem. One particular application of this gluing construction is to take (M_1, g_1) and (M_2, g_2) to be isometric and attach M_1 to M_2 along isometric ends. We will call this construction doubling along an end.

Remark 1 *(This remark was added during revisions.) D. Pollack observed that the proof of this theorem applies in a more general context, where (M_1, g_1) and (M_2, g_2) are noncompact, locally conformally flat manifolds with constant positive scalar curvature. The requirement that (M_1, g_1) and (M_2, g_2) are locally conformally flat allows one to use the same argument as in [CGS] to conclude that the ends are all asymptotically Delaunay. In this case one needs to assume that (in addition to the deformation which changes the asymptotic necksize of the chosen end of M_1) there is also a one-parameter family of deformations of (M_1, g_1) which translates the Delaunay*

neck of the chosen end of M_1. These deformations always exist in the case where (M_1, g_1) is a punctured sphere (see remark 2 and section 8 for more details).

One can think of this theorem as the scalar curvature analogue of a similar end to end gluing construction for surfaces of constant mean curvature in Euclidean space (see [R] and [MPPR]). In fact, most of the analysis is the same for the two constructions. This phenomenon has been widely noted (compare, e.g., [MPU2] and [KMP]), but not completely explained.

This theorem is also very much in the spirit of the results of Schoen in [S], of Mazzeo, Pollack and Uhlenbeck in [MPU1] and of Joyce in [J]. In all cases one constructs an approximate solution to the gluing problem, solves the linearized equation with uniform estimates, and solves the nonlinear problem using a fixed point theorem or an iteration technique. The constructions of Mazzeo and Pacard in [MP] and of Byde in [B] are similar in spirit, but use a different method, in that they solve boundary value problems on appropriate subdomains and then match Cauchy data.

I would like to thank F. Pacard for suggesting this problem. I would also like to thank D. Pollack, R. Mazzeo and F. Pacard for many useful suggestions as I was learning this subject.

2 Notation

In this section we establish some notation for the rest of the paper.

2.1 Notation for Delaunay Metrics

First we consider the Delaunay metrics. These can be written as

$$g_{\epsilon,T} = u_\epsilon^{\frac{4}{n-2}}(t+T)(dt^2 + d\theta^2)$$

on $\mathbb{R} \times S^{n-1}$. In the case $T = 0$ we will suppress it from the subscript. The function u_ϵ satisfies the ordinary differential equation

$$u'' - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0.$$

From this ODE one can show that u_ϵ is a periodic function uniquely determined by its minimal value ϵ (once we normalize u_ϵ so it achieves its minimum at $t = 0$). We will denote the period of u_ϵ by T_ϵ. As we will see in section 3, solutions of $L_{g_\epsilon}(u) = 0$ which lie outside a specific two-dimensional space satisfy a bound which (up to the change of variables $t \mapsto -t$) we can state as

$$|u(t,\theta)| \begin{cases} \leq & ce^t \quad t < 0 \\ \geq & ce^{-t} \quad t > 0. \end{cases}$$

For more discussion about the solutions to $L_{g_\epsilon} u = 0$, see section 3.

2.2 Notation for Everything Else in this Paper

Recall that we are starting with $(M_1 = S^n \backslash \{p_0, \ldots, p_{k_1-1}\}, g_1)$ and $(M_2 = S^n \backslash \{q_0, \ldots, q_{k_2-1}\}, g_2)$ two complete metrics with scalar curvature $n(n-1)$. We will assume that the p_j are mutually disjoint and that the q_j are mutually disjoint (we will allow, however, $p_j = q_{j'}$ for some j and j'). Let r_0 be small enough so that the balls $B_{r_0}(p_j)$ in the usual round metric are pairwise disjoint, and also so that the balls $B_{r_0}(q_j)$ in the usual round metric are pairwise disjoint. Let $M_1^c = S^n \backslash (\cup B_{r_0}(p_j))$ and $M_2^c = S^n \backslash (\cup B_{r_0}(q_j))$. Next fix two cutoff functions χ_1 and χ_2 such that

$$\chi_1(p) = \begin{cases} 0 & p \in M_1^c \\ 1 & p \in B_{r_0/2}(p_j) \backslash \{p_j\} \end{cases}$$

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and

$$\chi_2(p) = \begin{cases} 0 & p \in M_2^c \\ 1 & p \in B_{r_0/2}(q_j) \backslash \{q_j\}. \end{cases}$$

Inside $B_{r_0}(p_j)$ let $r_j(p)$ be the distance in the spherical metric to p_j and let $t_j = -\log(r_j/r_0)$. Similarly, in $B_{r_0}(q_j)$ let $\rho_j(p)$ be the distance is the spherical metric to q_j and let $\tau_j = -\log(\rho_j/r_0)$. Then with respect to these coordinates the asymptotics theorem (see [CGS] or [KMPS]) states that we can write the metric g_1 in $B_{r_0}(p_j)$ as

$$g_1 = (u_{1,j} + u_{\epsilon_j}(\cdot + T_j))^{\frac{4}{n-2}}(t_j, \theta_j)(dt_j^2 + d\theta_j^2)$$

where

$$\|u_{1,j}\|_{C^{2,\alpha}((\hat{t}_j-1, \hat{t}_j+1) \times S^{n-1})} = O(e^{-\hat{t}})$$

for $\hat{t}_j \geq 1$. Similarly, with respect to the coordinates (τ_j, θ_j), one can write

$$g_2 = (u_{2,j} + u_{\epsilon_j'}(\cdot + T_j'))^{\frac{4}{n-2}}(\tau_j, \theta_j)(d\tau_j^2 + d\theta_j^2)$$

where

$$\|u_{2,j}\|_{C^{2,\alpha}((\hat{\tau}_j-1, \hat{\tau}_j+1) \times S^{n-1})} = O(e^{-\hat{\tau}_j})$$

for $\hat{\tau}_j \geq 1$. We will see later that we can improve these estimates on $u_{1,0}$ and $u_{2,0}$ using conformal transformations of S^n. We will assume that $\epsilon_0 = \epsilon_0' = \epsilon$ is fixed throughout the rest of the paper.

3   Delaunay Metrics

In this section we will discuss some of the important features of the S^{n-1} invariant, complete, scalar curvature $n(n-1)$ metrics on $\mathbb{R} \times S^{n-1}$, which are known as Delaunay metrics. Most importantly, we discuss the spectral behavior of the operator $L_{g_\epsilon} = \Delta_{g_\epsilon} + n$ where g_ϵ is a Delaunay metric.

Recall that we can write the Delaunay metrics as

$$g_\epsilon = u_\epsilon^{\frac{4}{n-2}}(t)(dt^2 + d\theta^2)$$

where u_ϵ solves the ordinary differential equation

$$u'' - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0.$$

We remark that solutions to this ODE exist for all time because the equation has a conserved energy

$$H = (u_\epsilon')^2 - \frac{(n-2)^2}{4}u_\epsilon^2 + \frac{(n-2)^2}{4}u_\epsilon^{\frac{2n}{n-2}},$$

which would become unbounded if u_ϵ were to become unbounded. These metrics are uniquely determined by their singular set on S^n, the minimum value ϵ of the conformal factor and a translation parameter T. In fact, varying either parameter yields a one-parameter family of Delaunay metrics. Taking the derivative of this one parameter family, we obtain two linearly independent solutions to the Jacobi equation:

$$(\Delta_{g_\epsilon} + n)v_\epsilon^{0,\pm} = 0.$$

More precisely, we can write

$$v_\epsilon^{0,+} = \frac{d}{dT}\bigg|_{T=0} u_\epsilon(\cdot + T) \qquad v_\epsilon^{0,-} = \frac{d}{d\eta}\bigg|_{\eta=0} u_{\epsilon+\eta}.$$

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From this construction we see that $v_\epsilon^{0,\pm}$ are independent of θ, and thus they satisfy the ordinary differential equation

$$(v_\epsilon^{0,\pm})'' + 2\frac{u_\epsilon'}{u_\epsilon}(v_\epsilon^{0,\pm})' + nu_\epsilon^{\frac{4}{n-2}}v_\epsilon^{0,\pm} = 0.$$

We will normalize $v_\epsilon^{0,\pm}$ by choosing the initial conditions

$$v_\epsilon^{0,+}(0) = 1 \quad v_\epsilon^{0,-}(0) = 0 \quad (v_\epsilon^{0,+})'(0) = 0 \quad (v_\epsilon^{0,-})'(0) = 1.$$

Indeed, if we try to separate variables for a general solution v of

$$\Delta_{g_\epsilon} v + nv = 0,$$

we find that

$$v(t,\theta) = \sum v_j(t)\eta_j(\theta)$$

where η_j is the jth eigenfunction of $\Delta_{S^{n-1}}$ with eigenvalue λ_j (counted with multiplicity) and v_j satisfies the ordinary differential equation

$$v_j'' + 2\frac{u_\epsilon'}{u_\epsilon}v_j' + (n - \lambda_j)u_\epsilon^{\frac{4}{n-2}}v_j = 0. \tag{3}$$

The functions $v_\epsilon^{0,\pm}$ form a basis for the solution space to this ODE when $j = 0$. We will again choose a normalized pair of solutions to this ODE, $v_\epsilon^{j,+}$ and $v_\epsilon^{j,-}$, normalized so that

$$v_\epsilon^{j,+}(0) = 1 \quad v_\epsilon^{j,-}(0) = 0 \quad (v_\epsilon^{j,+})'(0) = 0 \quad (v_\epsilon^{j,-})'(0) = 1.$$

In fact, we can also find a basis for the solution space when $j = 1, \ldots, n$, again by taking explicit geometric deformations of the metric. To find these deformations, we use stereographic projection to write the Delaunay metric as

$$\hat{u}_\epsilon^{\frac{4}{n-2}} dx^2$$

where dx^2 is the standard Euclidean metric on $\mathbb{R}^n$ and $u > 0$ has a singularity at the origin. Now we can deform this metric by taking translates $a \mapsto \hat{u}_\epsilon(\cdot + a)$, so we obtain a Jacobi field by pulling back

$$\frac{d}{da_j}\Big|_{a=0} \hat{u}_\epsilon(\cdot + a) = e^{-t}(-\hat{v}_\epsilon^{0,+}(t) + \frac{n-2}{2}\hat{u}_\epsilon(t))\eta_j(\theta)$$

where η_j is the jth eigenfunction of $\Delta_{S^{n-1}}$ (see [KMPS]). Notice in particular that these functions all decay like e^{-t}. To find the other solution to equation (3), we first take the Kelvin transform of $\hat{u}_\epsilon(x) \mapsto |x|^{2-n}\hat{u}_\epsilon(x/|x|^2)$, translate as before and then take the Kelvin transform again. One can think of this deformation as a translation at infinity. Also, one can show that the Jacobi field associated to this deformation grows like e^t (again, see [KMPS]).

At this point we introduce the *indicial roots* of $L_{g_\epsilon} = \Delta_{g_\epsilon} + n$, denoted $\gamma_j(\epsilon)$. These are the exponential growth rates of the solutions to equation (3). From the above computations, one sees that $\gamma_0(\epsilon) = 0$ and $\gamma_j(\epsilon) = 1$ for $j = 1, \ldots, n$. By the maximum principle, $\gamma_j(\epsilon) > 1$ for $j > n$. Indeed, $\lambda_j \geq 2n$ for $j \geq n+1$, so the zero order term in this ODE is $(n - \lambda_j)\hat{u}_\epsilon^{\frac{4}{n-2}} \leq -n$. It is rather remarkable that one can compute $\gamma_j(\epsilon)$ for $j = 0, \ldots, n$ and that they are independent of ϵ, but the other indicial roots are quite hard to compute and probably depend on ϵ in some nontrivial way.

Another way to recover the indicial roots is to conjugate the operator $\Delta_{g_\epsilon} + n$ by an exponential function $e^{\delta t}$ and the Fourier-Laplace transform. Then one obtains a one (complex) parameter family of operators on a fixed function space, which varies analytically with the parameter. By the Analytic Fredholm Theorem, this family of operators has a meromorphic solution operator,

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and the indicial roots turn out to be the imaginary parts of the poles of this solution operator. In fact, they show that any solution u to

$$L_{g_\epsilon} u = 0$$

has an asymptotic expansion

$$u(t,\theta) \sim \sum_{j \geq 0} (a_{j,+} v_\epsilon^{j,+}(t) + a_{j,-} v_\epsilon^{j,-}(t)) \eta_j(\theta)$$

where η_j is the jth eigenfunction of $\Delta_{S^{n-1}}$ (counting multiplicity) and v_j are the particular solution of equation (3) listed above. In particular,

$$|v_\epsilon^{j,\pm}(t)| = O(e^{\pm \gamma_j(\epsilon)t}).$$

See [MPU2] for more about this approach.

A more thorough explanation of the indicial roots in the mean curvature setting occur in [MPPR], including an explanation of why they are called "indicial roots."

To sum up this discussion:

- $v_\epsilon^{0,+}$ is bounded and periodic and arises from translating the neck of the Delaunay metric towards the singularity

- $v_\epsilon^{0,-}$ is linearly growing and arises from changing the necksize of the Delaunay metric

- $v_\epsilon^{j,\pm}$ grow/decay like $e^{\mp t}$ for $j = 1, \ldots, n$ and both arise from translating the singular set of the Delaunay metric

- in fact, any solution u to $L_{g_\epsilon} u = 0$ which is L^2 orthogonal to $v_\epsilon^{0,\pm}$ on S^{n-1}-cross-sections has an expansion $u(t,\theta) \sim \sum_{j \geq 1} v_j(t) \eta_j(\theta)$ where $|v_j(t)| = O(e^{\pm \gamma_j(\epsilon)t})$ with $\gamma_1(\epsilon) = \gamma_2(\epsilon) = \cdots = \gamma_n(\epsilon) = 1$ and $1 < \gamma_{n+1}(\epsilon) \leq \gamma_{n+2}(\epsilon) \leq \cdots \to \infty$ (we have to exclude the $v_\epsilon^{0,\pm}$ terms because one of them grows linearly).

One can find rigorous proofs of the above facts in [MPU2], [MP] and [KMPS].

4   The Approximate Solution

In this section we construct the approximate solution $\tilde{g}_{R,\phi}$.

First we choose some $R = mT_{\epsilon_0}$ for some positive integer m and $\phi \in SO(n)$ and define M by

$$M = (M_1 \backslash B_{r_0 e^{-(T_0+R+1)}}(p_0)) \cup (M_2 \backslash B_{r_0 e^{-(T_0'+R+1)}}(q_0)) / \sim$$

where we identify (t_0, θ_0) with $(\tau_0, \phi\theta_0)$ if $t_0 \geq T_0 + R - 1$ and $\tau_0 \geq T_0' + R - 1$ and $t_0 + \tau_0 = T_0 + T_0' + 2R$. The balls $B_r(p_0)$ and $B_r(q_0)$ are balls in the standard round metric. We will let C_R be the cylinder $\{(t_0, \theta_0) : T_0 + R - 1 \leq t_0 \leq T_0 + R + 1\} \sim \{(\tau_0, \theta_0) : T_0' + R - 1 \leq \tau_0 \leq T_0' + R + 1\}$. We will also find it convenient in the following sections to define the extended cylinder

$$\hat{C}_R = (B_{r_0 e^{-T_0}}(p_0) \backslash B_{r_0 e^{-T_0-R+1}}(p_0)) \cup C_R \cup (B_{r_0 e^{-T_0'}}(q_0) \backslash B_{r_0 e^{-T_0'-R+1}}(q_0)),$$

parameterized by $(t, \theta) \in [-R, R] \times S^{n-1}$. The relationship between t and t_0 or τ_0 is given by $t = t_0 - R - T_0$ for $t < 0$ and $t = -\tau_0 + R + T_0'$ for $t > 0$. This relationship for between t and t_0 and τ_0 agrees with the identification of (t_0, θ) with $(\tau_0, \phi\theta)$ in C_R listed above.

Now we will define the metric $\tilde{g}_{R,\phi}$. First pick a cutoff function χ on M such that

$$\chi(p) = \begin{cases} 1 & p \in M_1 \backslash B_{r_0 e^{-(T_0+R-1)}}(p_0) \\ 0 & p \in M_2 \backslash B_{r_0 e^{-(T_0'+R-1)}}(q_0). \end{cases}$$

We define the metric $\tilde{g}_{R,\phi}$ by letting $\tilde{g}_{R,\phi} = g_1$ on $M_1 \backslash B_{r_0 e^{-(T_0+R-1)}}(p_0)$, letting $\tilde{g}_{R,\phi} = g_2$ on $M_2 \backslash B_{r_0 e^{-(T'_0+R-1)}}(q_0)$ and by letting

$$\tilde{g}_{R,\phi}(t_0,\theta_0) = (u_\epsilon(T_0+t_0)+\chi(t_0,\theta_0)u_{1,0}(t_0,\theta_0)+(1-\chi(t_0,\theta_0))u_{2,0}(T'_0+T_0+2R-t_0,\phi\theta_0))^{\frac{4}{n-2}}(dt_0^2+d\theta_0^2)$$

on C_R. The analysis below will often be independent of at least one of the parameters R and ϕ; in this case we will suppress the appropriate subscripts.

We denote the scalar curvature of $\tilde{g}_{R,\phi}$ by $\tilde{S}_{R,\phi}$. Outside of C_R, $\tilde{g}_{R,\phi}$ is either g_1 or g_2, and so $\tilde{S}_{R,\phi} = n(n-1)$ is these regions. A priori, we also have

$$\|u_{1,0}\|_{C^{2,\alpha}(C_R)} = O(e^{-R}),$$

(and a similar estimate for $u_{2,0}$). However, we can adjust g_1 and g_2 by conformal transformations as follows. The term $u_{1,0}$ has an asymptotic expansion near p_0 as

$$u_{1,0} \sim \sum_{j>1}(a_{j,+}v_\epsilon^{j,+}(t_0) + a_{j,-}v_\epsilon^{j,-}(t_0))\eta_j(\theta_0). \tag{4}$$

The functions $v_\epsilon^{j,\pm}$ correspond explicitly to translations of the origin or infinity once under the stereographic projection which sends p_0 to infinity. So change g_1 by the conformal motion which translates the origin by $(-a_{1,+},\ldots,-a_{n,+})$ and then by the conformal motion which translates infinity by $(-a_{1,-},\ldots,a_{n,-})$. This has the effect of eliminating the first n terms in the expansion (4), and so the new metric, which we will still call g_1, has an expansion of the form $(u_\epsilon + u_{1,0})^{\frac{4}{n-2}}(dt_0^2 + d\theta_0^2)$ where now

$$u_{1,0} \sim \sum_{j\geq n+1}(a_{j,+}v_\epsilon^{j,+}(t_0) + a_{j,-}v_\epsilon^{j,-}(t_0))\eta_j(\theta_0),$$

and so

$$\|u_{1,0}\|_{C^{2,\alpha}(C_R)} = O(e^{-\gamma_{n+1}(\epsilon)R}).$$

We can perform as similar adjustment to g_2 so that $\|u_{2,0}\|_{C^{2,\alpha}(C_R)} = O(e^{-\gamma_{n+1}(\epsilon)R})$. Notice we can only do this adjustment for one end of each of the M_i. The geometric effect of this adjustment is to translate the p_j and q_j (for $j \geq 0$) around so as to make the metrics g_1 and g_2 near p_0 and q_0 (respectively) closer to being Delaunay metrics. The above estimates imply the following lemma.

Lemma 2 When $(M, \tilde{g}_{R,\phi})$ is defined as above, $\tilde{S}_{R,\phi} = n(n-1) - \psi$ where ψ is compactly supported and $\|\psi\|_{C^{0,\alpha}(M)} = O(e^{-\gamma_{n+1}(\epsilon)R})$.

Proof: We only need to estimate $\tilde{S}_{R,\phi}$ in C_R. To this end, we first rewrite the conformal factor on C_R as $u_\epsilon(T_0+t_0)+\chi u_{1,0}(t_0,\theta_0)+(1-\chi)u_{2,0}(T'_0+T_0+2R-t_0,\phi\theta_0) = u_\epsilon(T_0+t_0)(1+v(t_0,\theta_0))$. If we plug $1+v$ into equation (1), we find

$$\begin{aligned}\left(\frac{n-2}{n-1}\right)\frac{\psi}{4}(1+v)^{\frac{n+2}{n-2}} &= \Delta_{g_\epsilon}(1+v) - \frac{n(n-2)}{4}(1+v) + \frac{n(n-2)}{4}(1+v)^{\frac{n+2}{n-2}}\\ &= \Delta_{g_\epsilon}(v) + nv + O(\|v\|_{C^{2,\alpha}}^2).\end{aligned}$$

The lemma now follows from the above bounds on $u_{1,0}$ and $u_{2,0}$, which imply similar bounds on v. $\blacksquare$

One way to rephrase the result of this lemma is to say that one can write the metric $\tilde{g}_{R,\phi}$ restricted to $\hat{C}_R$ (in the (t,θ) coordinates) as

$$\tilde{g}_{R,\phi} = (u_\epsilon(t) + v(t,\theta))^{\frac{4}{n-2}}(dt^2 + d\theta^2)$$

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where

$$|v(t, \theta)| = O\left(\frac{\cosh^{\gamma_{n+1}(\epsilon)} t}{\cosh^{\gamma_{n+1}(\epsilon)} R}\right).$$

Because of the above bounds on $\tilde{S}_{R,\phi}$ we will call the metric $\tilde{g}_{R,\phi}$ an approximate solution to our problem.

5   Linear Analysis

In this section we will develop the necessary linear analysis to find a uniformly bounded solution operator for the Jacobi operator $L_{\tilde{g}_{R,\phi}}$. We start the section by recalling some of the linear analysis for constant positive scalar curvature metrics in [MPU2] and then we construct a solution operator for $L_{\tilde{g}_{R,\phi}}$.

5.1   Linear Analysis for General Constant Scalar Curvature Metrics on Punctured Spheres

The growth properties for solutions of $L_{g_\epsilon}(u) = 0$ outlined above motivate the use of the following function spaces.

Definition 1 *On (M_i, g_i) we define $C_\delta^{l,\alpha}(M_i)$ to be the space of functions such that the norm*

$$\|u\|_{C_\delta^{l,\alpha}(M_i)} = \|u\|_{C^{l,\alpha}(M_i^c)} + \max_{0 \le j \le k_1 - 1} \sup_{\hat{t}_j \ge e^{-r_0} + 1} \|e^{-\delta t_j} u\|_{C^{l,\alpha}((\hat{t}_j - 1, \hat{t}_j + 1) \times S^{n-1})}$$

is finite. There is a similar definition for (M_2, g_2). For the approximate solution $(M, \tilde{g}_{R,\phi})$ we will need to adjust this definition as follows. Recall that we can write

$$M = M_1^c \cup (\cup_1^{k_1 - 1} B_{r_0}(p_j) \backslash \{p_j\}) \cup M_2^c \cup (\cup_1^{k_2 - 1} B_{r_0}(q_j) \backslash \{q_j\}) \cup \hat{C}_R.$$

Then we define $C_\delta^{l,\alpha}(M)$ to be the space of functions such that the norm

$$\begin{aligned}\|u\|_{C_\delta^{l,\alpha}(M)} &= \|u\|_{C^{l,\alpha}(M_1^c)} + \|u\|_{C^{l,\alpha}(M_2^c)} + \max_{1 \le j \le k_1 - 1} \sup_{\hat{t}_j \ge e^{-r_0} + 1} \|e^{-\delta t_j} u\|_{C^{l,\alpha}((\hat{t}_j - 1, \hat{t}_j + 1) \times S^{n-1})} \\ &\quad + \max_{1 \le j \le k_2 - 1} \sup_{\hat{\tau}_j \ge e^{-r_0} + 1} \|e^{-\delta \tau_j} u\|_{C^{l,\alpha}((\hat{\tau}_j - 1, \hat{\tau}_j + 1) \times S^{n-1})} + \sup_{|\hat{t}| \le R - 1} \|\frac{\cosh^\delta R}{\cosh^\delta t} u\|_{C^{l,\alpha}([\hat{t} - 1, \hat{t} + 1] \times S^{n-1})}\end{aligned}$$

is finite. We also say (M_i, g_i) is unmarked nondegenerate if $L_{g_i} u = 0$ does not admit solutions $u \in C_\delta^{2,\alpha}(M_i)$ for any $\delta < -1$.

Functions in $C_\delta^{l,\alpha}(M_i)$ can grow at most like $e^{\delta t_j}$ on the end E_j. We remark that for $\delta \le 1$ the only solutions of $\Delta_{g_\epsilon} v + nv = 0$ with $v \in C_\delta^{2,\alpha}(\mathbb{R} \times S^{n+1})$ are linear combinations of $v_\epsilon^{j,+}$ and $v_\epsilon^{j,-}$ for $j = 0, \ldots, n$, each of which is either bounded and periodic or unbounded on at least one end. So the Delaunay metrics are unmarked nondegenerate. We also remark that the function space $C_\delta^{l,\alpha}(M)$ is the same space of functions as if we had not weighted the middle cylinder, but it has a different norm. This difference in norms will become important later when we want uniform bounds on a solution operator. We further remark that the notion of unmarked nondegenerate is weaker than the notion of marked nondegenerate, which requires that $L_{g_i} u = 0$ does not have any solutions where $u \in C_\delta^{2,\alpha}(M_i)$ with $\delta < 0$.

In order to find a function space on which L_{g_i} has suitable mapping properties we will need the following definition.

Definition 2 *The deficiency space W_{g_i} of (M_i, g_i) is the span of all the functions $\chi_i v_{\epsilon_j}^{i,\pm}$, where $1 \le j \le k_i$ (so the sum runs over all the ends of M_i) and $0 \le i \le n$.*

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Notice that W_{g_i} is a vector space of dimension $k_i(2n+2)$ and it has a basis $\{\chi_i v_{\epsilon_j}^{i,\pm}\}$ which only depends on the metric g_i and the choice of cutoff function χ_i. We will use this basis to give W_{g_i} the Euclidean norm. We also remark that this is the proper deficiency space to use to parameterize the unmarked moduli space, meaning that one fixes the cardinality but not the position of the singular set. For the marked moduli space (fixing both the cardinality and the position of the singular set) one should work with a smaller deficiency space which only incorporates the Jacobi fields arising from translating the necks of the Delaunay metrics along their axes and changing their necksizes.

Remark 2 *Let E_j be an end of M_1 corresponding to the puncture point p_j. It turns out the we can use particular conformal Killing fields on the sphere to show that for any end E_j there is always a Jacobi field of g_i which is asymptotic to $v_{\epsilon_j}^{0,+}$ along E_j. To see this, consider stereographic projection sending p_j to ∞ composed with a dilation about the origin. This provides a one-parameter family of scalar curvature $n(n-1)$ metrics on M_1 which translate the Delaunay neck on E_j. Taking the infinitesimal generator of this family we obtain a Jacobi field asymptotic to $v_\epsilon^{0,+}$. Similar Jacobi fields exist on M_2. These Jacobi fields are also in $C_1^{2,\alpha}(M_i)$. This seems to be a special property of spheres, as one cannot in general find such conformal Killing fields on arbitrary compact manifolds with positive scalar curvature. In the mean curvature case the corresponding Jacobi fields arise from global translations of the surface.*

A similar Linear Decomposition result to the one stated below appears as Lemma 4.18 of [MPU2], as stated for weighted Sobolev spaces and exactly constant scalar curvature metrics. The result below is essentially the next term in the asymptotic expansion; see Proposition 4.15 of [MPU2]. The proof for weighted Hölder spaces is nearly identical and really only requires that the ends are asymptotically Delaunay.

Proposition 3 (Mazzeo, Pollack, Uhlenbeck, 1996) *Let $\delta \in (1, \inf \gamma_{n+1}(\epsilon_j))$. If $u \in C_\delta^{2,\alpha}(M_i)$, $f \in C_{-\delta}^{0,\alpha}(M_i)$ and $L_{g_i}u = f$ then $u \in W_{g_i} \oplus C_{-\delta}^{2,\alpha}(M_i)$.*

Suppose g_i is a unmarked nondegenerate metric on M_i. Then for $\delta \in (1, \inf(\gamma_{n+1}(\epsilon_j))$

$$L_{g_i} : C_{-\delta}^{2,\alpha}(M_i) \to C_{-\delta}^{0,\alpha}(M_i)$$

is injective, which in turn implies

$$L_{g_i} : C_\delta^{2,\alpha}(M_i) \to C_\delta^{0,\alpha}(M_i)$$

is surjective. If we combine this with the Linear Decomposition result in proposition 3 then we see that
$$L_{g_i} : W_{g_i} \oplus C_{-\delta}^{2,\alpha}(M_i) \to C_{-\delta}^{0,\alpha}(M_i)$$
is surjective. We will call the kernel of this map B_{g_i}, the *bounded null space* of L_{g_i}. Mazzeo, Pollack and Uhlenbeck ([MPU2]) show that if M_i has k_i ends and g_i is unmarked nondegenerate then B_{g_i} is $k_i(n+1)$-dimensional (in general B_{g_i} could contain a space of exponentially decaying functions of some unknown dimension). From this reasoning one can see (using the Implicit Function Theorem) that near an unmarked nondegenerate point the moduli space of such metrics has the structure of a real analytic manifold of dimension $k_i(n+1)$.

5.2 Solvability of the Linear Problem

To construct the deficiency space $W_{\tilde{g}_{R,\phi}}$ we take cutoffs of the Jacobi fields $v_\epsilon^{i,\pm}$ from the model Delaunay metrics arising from $p_1 \ldots p_{k_1-1}$ and $q_1, \ldots q_{k_2-1}$. Notice we do not include p_0 and q_0. Again, we will use the basis formed by $\{\chi_1 v_{\epsilon_j}^{i,\pm}, \chi_2 v_{\epsilon_j}^{i,\pm}\}$, which induces the Euclidean norm on $W_{\tilde{g}_{R,\phi}}$.

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Recall that the Jacobi operator $L_{\tilde{g}+R,\phi}$ is given by

$$L_{\tilde{g}R,\phi} = \Delta_{\tilde{g}R,\phi} + n + \left(\frac{n-2}{n-1}\right)\frac{\psi}{4},$$

which is a perturbation of $\Delta_{\tilde{g}R,\phi} + n$ where the perturbation is compactly supported and globally of size $O(e^{-\gamma_{n+1}(\epsilon)R})$.

Proposition 4 *Suppose both g_i are unmarked nondegenerate and there exists a one-parameter family of scalar curvature $n(n-1)$ metrics g_t on M_1 such that the asymptotic necksize of the end at p_0 with respect to to g_t is $\epsilon + t$. Then for $\delta \in (1, \inf\{\gamma_{n+1}(\epsilon_j), \gamma_{n+1}(\epsilon'_j)\})$ there exists an $R_0 > 0$ such that for $R \geq 0$ one can find an operator*

$$G_{R,\phi} : C^{0,\alpha}_{-\delta}(M) \to W_{\tilde{g}R,\phi} \oplus C^{2,\alpha}_{-\delta}(M)$$

such that $u = G_{R,\phi}(f)$ solves the equation $L_{\tilde{g}R,\phi}(u) = f$ and $\|u\|_{W_{\tilde{g}R,\phi}\oplus C^{2,\alpha}_{-\delta}(M)} \leq c\|f\|_{C^{0,\alpha}_{-\delta}(M)}$ where c is independent of R and ϕ.

The idea behind this proof was communicated to me by F. Pacard.
Proof: We wish to solve the equation

$$L_{\tilde{g}R,\phi}(u) = f.$$

To this end, first let $u_1 + v_1 \in W_{g_1} \oplus C^{2,\alpha}_{-\delta}(M_1)$ solve

$$L_{g_1}(u_1 + v_1) = \chi f.$$

Such a solution exists because g_1 is unmarked nondegenerate. Moreover, we have the estimate

$$\|u_1\|_{W_{g_1}} + \|v_1\|_{C^{2,\alpha}_{-\delta}(M_1)} \leq c_1\|\chi f\|_{C^{0,\alpha}_{-\delta}(M_1)}. \tag{5}$$

In $B_{r_0}(p_0)$ (the standard spherical ball), $u_1(t_0,\theta_0) \sim \sum_{i,\pm} \alpha_{i,\pm} v^{i,\pm}_\epsilon(t_0)$. Now choose $\Phi_1 \in B_{g_1}$ such that $|\Phi_1(t_0,\theta_0) + \sum \alpha_{i,\pm} v^{i,\pm}_\epsilon(t_0)| = O(e^{-\gamma_{n+1}(\epsilon)t_0})$. We can choose such a Φ_1 because of the existence of the Jacobi fields in remark 2 and because of the assumption that there is a one-parameter family of metrics g_t on M_1 such that $g_0 = g_1$ and the asymptotic necksize of the end at p_0 with respect to g_t is $\epsilon + t$. Thus $L_{g_1}(u_1 + v_1 + \Phi_1) = \chi f$ and

$$|u_1(t_0,\theta_0) + v_1(t_0,\theta_0) + \Phi_1(t_0,\theta_0)| \leq 2c_1\|\chi f\|_{C^{0,\alpha}_{-\delta}(M_1)}e^{-\delta t_0} \tag{6}$$

for $(e^{-t_0},\theta) \in B_{r_0}(p_0)$. We also have the estimate

$$\|\Phi_1\|_{W_{g_1}\oplus C^{2,\alpha}_{-\delta}(M_1)} \leq c_1\|\chi f\|_{C^{0,\alpha}_{-\delta}(M_1)}. \tag{7}$$

Similarly we let $u_2 + v_2 \in W_{g_2} \oplus C^{2,\alpha}_{-\delta}(M_2)$ solve

$$L_{g_2}(u_2 + v_2) = (1 - \chi)f$$

with the estimate

$$\|u_2\|_{W_{g_2}} + \|v_2\|_{C^{2,\alpha}_{-\delta}(M_2)} \leq c_2\|f\|_{C^{0,\alpha}_{-\delta}(M_2)}. \tag{8}$$

This time we cannot cancel the nondecaying part of $u_2 + v_2$ on E_2. Instead, let $\beta_{i,\pm}$ be such that $|u_2(\tau_0,\theta_0) - \sum \beta_{i,\pm} v^{i,\pm}_\epsilon(\tau_0)| = O(e^{-\gamma_{n+1}(\epsilon)\tau_0})$ and let $\Phi_2 \in B_{g_1}$ be such that $|\Phi_2(t_0,\theta_0) - \sum \beta_{i,\pm} v^{i,\pm}_\epsilon(t_0)| = O(e^{-\gamma_{n+1}(\epsilon)t_0})$ (recall that the relationship between t_0 and τ_0 in $\hat{C}_R$ is given by $t_0 + \tau_0 = T_0 + T'_0 + 2R$). This time the salient estimates are

$$\|\Phi_2\|_{W_{g_1}\oplus C^{2,\alpha}_{-\delta}(M_1)} \leq c_2\|(1 - \chi)f\|_{C^{0,\alpha}_{-\delta}(M_2)} \tag{9}$$

and

$$\|L_{\tilde{g}_{R,\phi}}(\Phi_2 + u_2)\|_{C^{0,\alpha}(C_R)} \le c_3 \|f\|_{C^{0,\alpha}_{-\delta}(M)} e^{-\delta R}. \tag{10}$$

This last estimate is a straightforward calculation using the facts that Φ_2 is a Jacobi field of g_1, $u_2 \in W_{g_2}$, and both g_1 and g_2 (and hence $\tilde{g}_{R,\phi}$) are $C^{2,\alpha}$-close to being Delaunay metrics in C_R.

Now choose cutoff functions η_1 and η_2 such that

$$\eta_1(p) = \begin{cases} 1 & p \in (M_1 \backslash B_{r_0 e^{-(T_0 + R - 1)}}(p_0)) \cup C_R \cup (B_{r_0 e^{-(T_0' + 2)}}(q_o) \backslash B_{r_0 e^{-(T_0' + R - 1)}}(q_0)) \\ 0 & p \in M_2 \backslash B_{r_0 e^{-(T_0' + 1)}}(q_0) \end{cases}$$

and

$$\eta_2(p) = \begin{cases} 1 & p \in (M_2 \backslash B_{r_0 e^{-(T_0' + R - 1)}}(q_0)) \cup C_R \cup (B_{r_0 e^{-(T_0 + 2)}}(p_0) \backslash B_{r_0 e^{-(T_0 + R - 1)}}(p_0)) \\ 0 & p \in M_1 \backslash B_{r_0 e^{-(T_0 + 1)}}(p_0), \end{cases}$$

and define

$$u + v = \hat{G}_{R,\phi}(f) = \eta_1(u_1 + v_1 + \Phi_1) + \eta_2 v_2 + (1 - \chi)u_2 + \chi\Phi_2.$$

We will complete the proof of this proposition by showing

- $\|u + v\|_{W_{\tilde{g}_{R,\phi}} \oplus C^{2,\alpha}_{-\delta}(M)} \le c\|f\|_{C^{0,\alpha}_{-\delta}(M)}$ and

- $\|L_{\tilde{g}_{R,\phi}}(u + v) - f\|_{C^{0,\alpha}_{-\delta}(M)} \le \tilde{c}\|f\|_{C^{0,\alpha}_{-\delta}(M)} e^{-\tilde{\gamma}R}$ for some $\tilde{\gamma} > 0$

where c and $\tilde{c}$ are independent of R. The above estimates show $\hat{G}_{R,\phi}$ is uniformly bounded and $L_{\tilde{g}_{R,\phi}} \circ \hat{G}_{R,\phi} = \mathrm{Id} + O(e^{-\tilde{\gamma}R})$ and the statement of the proposition follows by a simple perturbation argument. The first estimate follows immediately from the estimates (5), (7), (8) and (9). For the second estimate, notice $L_{\tilde{g}_{R,\phi}}(u + v) - f \ne 0$ only where $\nabla\chi$, $\nabla\eta_1$ or $\nabla\eta_2$ are nonzero. The region where $\nabla\eta_1 \ne 0$ corresponds to $T_0' + 1 \le \tau_0 \le T_0' + 2$, or $R - 2 \le t \le R - 1$. In this region

$$\begin{aligned} |L_{\tilde{g}_{R,\phi}}(u + v)(t,\theta) - f(t,\theta)| &= |L_{\tilde{g}_{R,\phi}}(\eta_1(u_1 + v_1 + \Phi_1)(t,\theta)| \\ &\le \|L_{\tilde{g}_{R,\phi}}\| \cdot |u_1(t,\theta) + v_1(t,\theta) + \Phi_1(t,\theta)| \\ &= O(e^{-2\delta R})\|f\|_{C^{0,\alpha}_{-\delta}(M)}. \end{aligned}$$

One can similarly estimate $L_{\tilde{g}_{R,\phi}}(u + v) - f$ in the regions $\nabla\chi \ne 0$ and $\nabla\eta_2 \ne 0$. ∎

Whenever we wish to solve the equation $L_{\tilde{g}_{R,\phi}}(u) = f$ where f decays at some exponential rate we will always use the solution operator we constructed in the above proposition. In general, one can find many solution operators for $L_{\tilde{g}_{R,\phi}}$ but most of them will not be uniformly bounded in R, as $G_{R,\phi}$ is.

6 Nonlinear Analysis

There are two parts to the nonlinear part of this problem. First we construct explicit deformations of the metric $\tilde{g}_{R,\phi}$, parameterized by a small ball about the origin in $W_{\tilde{g}_{R,\phi}}$. These deformed metrics will not be conformal to $\tilde{g}_{R,\phi}$, but their conformal class will always lie close to that of $\tilde{g}_{R,\phi}$ in the Gromov-Hausdorff topology (see below). One can think of this step as an exponential map from a subspace of the tangent space of all metrics to the space of metrics itself. Finally, we use the solution operator $G_{R,\phi}$ to to build a contraction from a small ball in $W_{\tilde{g}_{R,\phi}} \oplus C^{2,\alpha}_{-\delta}(M)$ to itself. The fixed point of this contraction will be the solution to our nonlinear problem.

6.1 The Geometric Deformations

In this section we define geometric deformations of the approximate solution $\tilde{g}_{R,\phi}$ corresponding to elements in $W_{\tilde{g}_{R,\phi}}$.

First recall that we can write $w \in W_{\tilde{g}_{R,\phi}}$ as

$$w = \chi_1 \sum_{j=1}^{k_1-1} (\alpha_j^{i,+} v_{\epsilon_j}^{i,+} + \alpha_j^{i,-} v_{\epsilon_j}^{i,-}) \eta_i + \chi_2 \sum_{j=1}^{k_2-1} (\beta_j^{i,+} v_{\epsilon_j'}^{i,+} + \beta_j^{i,-} v_{\epsilon_j'}^{i,-}) \eta_i$$

where $v_{\epsilon_j}^{i,\pm}$ and $v_{\epsilon_j'}^{i,\pm}$ are the Jacobi fields for g_{ϵ_j} and $g_{\epsilon_j'}$ described in section 3. For each end of M_1, let $\hat{g}_j$ be the deformed Delaunay metric obtained as follows: First replace $u_{\epsilon_j}(\cdot + T_j)$ with $u_{\epsilon_j + \alpha_j^{0,-}}(\cdot + T_j + \alpha_j^{0,+})$. Then transfer to $\mathbb{R}^n$ via stereographic projection and translate the origin by $(\alpha_j^{1,+}, \ldots, \alpha_j^{n,+})$ and translate infinity by $(\alpha_j^{1,-}, \ldots, \alpha_j^{n,-})$. Then pull the result back by the inverse of stereographic projection. The result is a new Delaunay metric $(\hat{u}_j)^{\frac{4}{n-2}} (dt_j^2 + d\theta^2)$. Similarly, we obtain a new Delaunay metric $(\hat{u}_{j'})^{\frac{4}{n-2}} (d\tau_j^2 + d\theta^2)$ on each end of M_2.

Given $w \in W_{\tilde{g}_{R,\phi}}$ in the above form, we will define a new metric $\tilde{g}_{R,\phi}(w)$ as follows. First let M^c be $M \backslash ((\cup_1^{k_1-1} B_{r_0}(p_j)) \cup (\cup_1^{k_2-1} B_{r_0}(q_j)))$ where the balls are taken with respect to the standard spherical metric. Let $\hat{\chi}$ be a cutoff function such that

$$\hat{\chi}(p) = \left\{ \begin{array}{ll} 1 & p \in (\cup_1^{k_1-1} B_{r_0/2}(p_j) \backslash \{p_j\}) \cup (\cup_1^{k_2-1} B_{r_0/2}(q_j) \backslash \{q_j\}) \\ 0 & p \in M^c. \end{array} \right.$$

We will let $\tilde{g}_{R,\phi}(w) = \tilde{g}_{R,\phi}$ on M^c. In $B_{r_0}(p_j) \backslash \{p_j\}$ we define $\tilde{g}_{R,\phi}$ in the local cylindrical coordinates (see section 2) by

$$\tilde{g}_{R,\phi}(w) = (u_{1,j} + (1-\chi) u_{\epsilon_j}(\cdot + T_j) + \chi \hat{u}_j(\cdot + T_j))^{\frac{4}{n-2}} (dt_j^2 + d\theta_j^2).$$

We define $\tilde{g}_{R,\phi}(w)$ in $B_{r_0}(q_j) \backslash \{q_j\}$ similarly.

In defining the deformed metric $\tilde{g}_{R,\phi}(w)$ we induce new perturbations in the scalar curvature, compactly supported in $B_{r_0}(p_j) \backslash \{p_j\}$ and $B_{r_0}(q_j) \backslash \{q_j\}$. But this perturbation is small, as one can see by taking a Taylor expansion of the scalar curvature as given in equation (1). The proof of the following lemma is a straightforward computation and left to the reader.

Lemma 5 *For $\delta \in (1, \inf\{\gamma_{n+1}(\epsilon_j), \gamma_{n+1}(\epsilon_j')\})$, the scalar curvature of $\tilde{g}_{R,\phi}(w)$ is given by $\tilde{S}_{R,\phi}(w) = n(n-1) - \psi - \hat{\psi}(w)$ where $\|\hat{\psi}(w)\|_{C_{-\delta}^{0,\alpha}(M)} = O(|w|)$, but not $o(|w|)$. Moreover, $\hat{\psi}(w)$ is compactly supported in $(\cup_{j=1,\ldots,k_1-1}(B_{r_0}(p_j) \backslash B_{r_0/2}(p_j))) \cup (\cup_{j=1,\ldots,k_2-1}(B_{r_0}(q_j) \backslash B_{r_0/2}(q_j)))$.*

Thus in particular the perturbation in the scalar curvature decays exponentially, so the nonlinear operator is well-behaved.

Notice that the metric $\tilde{g}_{R,\phi}(w)$ will not be conformal to $\tilde{g}_{R,\phi}$, but for small w the metrics will be close on large compact sets. More precisely, given a compact set $\Omega \subset M$ and $\nu > 0$ there is an $\eta > 0$ such that $\|\tilde{g}_{R,\phi}(w) - \tilde{g}_{R,\phi}\|_{C^{0,\alpha}(\Omega)} \leq \nu$ for $|w| \leq \eta$. In other words, $\tilde{g}_{R,\phi}(w)$ and $\tilde{g}_{R,\phi}$ are close in the Gromov-Hausdorff topology. This is a more precise way of saying that the conformal class of $\tilde{g}_{R,\phi}(w)$ (and hence that of the constant scalar curvature metric we will construct) is close to the conformal class of $\tilde{g}_{R,\phi}$.

6.2 Solving the Gluing Problem with a Contraction

We wish to solve the nonlinear equation 1, which we will restate here with $S_g = n(n-1) - \psi$ and $S_{g'} = n(n-1)$:

$$0 = \Delta_{\tilde{g}_{R,\phi}} u - \frac{n(n-2)}{4} u + \left(\frac{n-2}{n-2} \right) \frac{\psi}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = L_{\tilde{g}_{R,\phi}} u + Q_{R,\phi}(u).$$

Above we have written a Taylor series expansion for the scalar curvature operator, where

$$L_{\tilde{g}R,\phi} = \Delta_{\tilde{g}R,\phi} + n + \left(\frac{n-2}{n-1}\right)\frac{\psi}{4}$$

and

$$Q_{R,\phi}(u) = \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} - \frac{n(n+2)}{4}u$$

incorporates all the second and higher order terms of the Taylor series. In particular,

$$Q_{R,\phi}(0) = 0 \qquad \nabla Q_{R,\phi}(0) = 0,$$

and so

$$\|Q_{R,\phi}(u)\|_{C^{0,\alpha}_{-\delta}(M)} \le C_Q\|u\|^2_{W_{\tilde{g}R,\phi}\oplus C^{2,\alpha}_{-\delta}(M)}$$

for some C_Q.

The above analysis deserves some comment before we continue. Firstly, we decompose u as $w + v$ where $w \in W_{\tilde{g}R,\phi}$ and $v \in C^{2,\alpha}_{-\delta}(M)$. We make sense of the scalar curvature operator applied to u as follows: first deform $\tilde{g}_{R,\phi}$ to $\tilde{g}_{R,\phi}(w)$ as in section 6.1 and then let the new metric g be given by $g = (1+v)^{\frac{4}{n-2}}\tilde{g}_{R,\phi}(w)$. The scalar curvature of the new metric g is the scalar curvature operator applied to u. Finding $u = v + w \in W_{\tilde{g}R,\phi} \oplus C^{2,\alpha}_{-\delta}(M)$ such the the scalar curvature of $g = (1+v)^{\frac{4}{n-2}}\tilde{g}_{R,\phi}(w)$ is $n(n-1)$ is equivalent to solving the equation $L_{\tilde{g}R,\phi}(u) = -Q_{R,\phi}(u)$. Moreover, both of these operators are well defined acting on $W_{\tilde{g}R,\phi} \oplus C^{2,\alpha}_{-\delta}(M)$ and map into $C^{0,\alpha}_{-\delta}(M)$. To see that $Q_{R,\phi}(u)$ decays exponentially, recall that that the new metric g restricted to the ends is still conformal to a Delaunay metric with a conformal factor that is exponentially close to 1. Notice that $Q_{R,\phi}(u)$ contains terms from the conformal factor v and from the perturbation term $\hat{\psi}(w)$.

The existence part of theorem 1 follows immediately from the following proposition.

Proposition 6 *The map*

$$K_{R,\phi} : W_{\tilde{g}R,\phi} \oplus C^{2,\alpha}_{-\delta}(M) \to W_{\tilde{g}R,\phi} \oplus C^{2,\alpha}_{-\delta}(M)$$

given by

$$K_{R,\phi}(u) = -G_{R,\phi}(Q_{R.\phi}(u))$$

is a contraction on sufficiently a small ball centered at the origin, and thus it has a unique fixed point.

Proof: First we estimate

$$\begin{aligned}\|K_{R,\phi}(u)\|_{W_{\tilde{g}R,\phi}\oplus C^{2,\alpha}_{-\delta}(M)} &\le & \|G_{R,\phi}\| \cdot \|Q_{R,\phi}(u)\|_{C^{0,\alpha}_{-\delta}(M)}\\ &\le & C_Q\|G_{R,\phi}\| \cdot \|u\|^2_{W_{\tilde{g}R,\phi}\oplus C^{2,\alpha}_{-\delta}(M)},\end{aligned}$$

which shows that $K_{R,\phi}$ maps a small ball to itself. Also,

$$\begin{aligned}\|K_{R,\phi}(u_1) - K_{R,\phi}(u_2)\|_{W_{\tilde{g}R,\phi}\oplus C^{2,\alpha}_{-\delta}(M)} &\le & C_Q\|G_{R,\phi}\| \cdot \|u_1 - u_2\|^2_{W_{\tilde{g}R,\phi}\oplus C^{2,\alpha}_{-\delta}(M)}\\ &\le & 2C_Q\|G_{R,\phi}\| \max\{\|u_1\|, \|u_2\|\}\|u_1 - u_2\|\\ &\le & \frac{1}{2}\|u_1 - u_2\|.\end{aligned}$$

$\blacksquare$

If $u = w + v$ is the unique fixed point of $K_{R,\phi}$ then we define $g_{R,\phi}$ to the solution to our nonlinear gluing problem:

$$g_{R,\phi} = (1+v)^{\frac{4}{n-2}}\tilde{g}_{R,\phi}(w).$$

7 Nondegeneracy of the Solution

In this section we will first prove some preliminary lemmas and then show that for R sufficiently large the metric $g_{R,\phi}$ is unmarked nondegenerate. The preliminary lemmas in section 7.1 seem to be interesting in their own right.

7.1 Some Preliminary Lemmas

In order to prove that $g_{R,\phi}$ is unmarked nondegenerate for sufficiently large R we will need the following lemmas.

Lemma 7 *Suppose $v \in B_{g_1}$ decays like $e^{-\delta_j t_j}$ near all p_j for some $\delta_j > 1$ except p_0. Then $v \sim a v_\epsilon^{0,+}$ near p_0 for some $a \in \mathbb{R}$. Moreover, if there exists $w \in B_{g_1}$ with $w \sim v_\epsilon^{0,-}$ near p_0 then v decays at least like $e^{-\delta t_0}$ for some $\delta > 1$ near p_0.*

Proof: We know that $v \sim \sum(\alpha_{i,+} v_\epsilon^{i,+} + \alpha_{i,-} v_\epsilon^{i,-})\eta_i$ near p_0, so suppose $\alpha_{0,-} \neq 0$. Recall from remark 2 that we have a Jacobi field $w \in B_{g_1}$ such that $w \sim v_\epsilon^{0,+}$ near p_0 and such that $w \in C_1^{2,\alpha}(M_1)$. Then

$$
\begin{aligned}
0 &= \lim_{r \to 0} \int_{M_1 \setminus (\cup B_r(p_j))} w L_{g_1} v - v L_{g_1} w \\
&= \lim_{r \to 0} \int_{\partial M_1 \setminus (\cup B_r(p_j))} w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \\
&= \lim_{r \to 0} \Big[\int_{t_0 = -\log r} \alpha_{0,-} (v_\epsilon^{0,+}(v_\epsilon^{0,-})' - v_\epsilon^{0,-}(v_\epsilon^{0,+})') + O(r^{\gamma_{n+1}(\epsilon)}) + O(\sup_{j \geq 1} r^{\delta_j}) \Big].
\end{aligned}
$$

But this last term is just $\alpha_{0,-}$ times the Wronskian of $v_\epsilon^{0,+}$ and $v_\epsilon^{0,-}$, which one can write explicitly as $v_\epsilon^{0,+}(v_\epsilon^{0,-})' - v_\epsilon^{0,-}(v_\epsilon^{0,+})' = e^{u_\epsilon^2 - u_\epsilon^2(0)}$, which is bounded away from zero. The proof of the remaining case of this lemma uses an identical argument. ∎

We will also need a lemma regarding Delaunay metrics on finite cylinders. Before we can state this lemma we need to define the following function space.

Definition 3 *The function space $C_\delta^{l,\alpha}([-T,T] \times S^{n-1})$ is defined to be the space of functions such that*

$$
\|u\|_{C_\delta^{l,\alpha}} = \sup_{|\hat{t}| \leq T-1} \| \left(\frac{\cosh^\delta(T)}{\cosh^\delta(t)} \right) \cdot u \|_{C^{l,\alpha}([\hat{t}-1, \hat{t}+1] \times S^{n-1})}
$$

is finite.

Lemma 8 *Pick $\delta \in (1, \gamma_{n+1}(\epsilon))$. Then there exists an operator*

$$
H_{T,\epsilon} : C_{-\delta}^{0,\alpha}([-T,T] \times S^{n-1}) \to C_{-\delta}^{2,\alpha}([-T,T] \times S^{n-1})
$$

such that $u = H_{T,\epsilon}(f)$ solves $L_{g_\epsilon}(u) = f$ Moreover, $H_{T,\epsilon}$ is uniformly bounded in T.

Notice we do not say anything about the boundary values of $H_{T,\epsilon}(f)$, other than that they are bounded by $\|H\| \cdot \|f\|_{C_{-\delta}^{2,\alpha}([-T,T] \times S^{n-1})}$.

Proof: First choose a cutoff function β on $[-T,T] \times S^{n-1}$ such that

$$
\beta(t,\theta) = \begin{cases} 1 & t \leq -1 \\ 0 & t \geq 1. \end{cases}
$$

Next let $u_1 \in C_{-\delta}^{2,\alpha}([-T,\infty) \times S^{n-1})$ solve $L_{g_\epsilon}(u_1) = \beta f$. We can find such a solution because g_ϵ is unmarked nondegenerate and we can use $v_\epsilon^{i,\pm}$ to eliminate the part of u_1 which grows at a

rate of e^t or less. Similarly let $u_2 \in C^{2,\alpha}_\delta((-\infty, T] \times S^{n-1})$ solve $L_{g_\epsilon}(u_2) = (1 - \beta)f$. If we let $\hat{H}_{T,\epsilon}(f) = \beta u_1 + (1 - \beta)u_2$ then the lemma follows from a perturbation argument as in the proof of proposition 4.
■

7.2 The Nondegeneracy

In this section we complete the proof of theorem 1 by showing that $g_{R,\phi}$ is unmarked nondegenerate for R sufficiently large. We will argue by contradiction, assuming that for some sequence $R_l \to \infty$ the metrics $g_l = g_{R_l,\phi}$ are unmarked degenerate.

Thus we can find $\delta_l > 1$ and $0 \neq u_l \in C^{2,\alpha}_{-\delta_l}(M)$ such that $L_{g_l}(u_l) = 0$. We will normalize u_l so that

$$\sup_M \rho_l^{-1}|u_l| = 1$$

where ρ_l is a positive weighting function we will define in the next paragraph.

First choose for $\delta \in (1, \inf\{\gamma_{n+1}(\epsilon_j), \gamma_{n+1}(\epsilon'_j)\})$ and recall that for each l we can decompose M as

$$M = M_1^c \cup M_2^c \cup (\cup_1^{k_1-1} B_{r_0}(p_j)\backslash\{p_j\}) \cup (\cup_1^{k_2-1} B_{r_0}(q_j)\backslash\{q_j\}) \cup \hat{C}_{R_l}.$$

Then we define the weighting function ρ_l by

$$\rho_l(p) = \begin{cases} 1 & p \in M_i^c \\ e^{-\delta t_j} & p \in B_{r_0/2}(p_j)\backslash\{p_j\} \text{ for } j = 1\ldots k_1 - 1 \\ e^{-\delta \tau_j} & p \in B_{r_0/2}(q_j)\backslash\{q_j\} \text{ for } j = 1\ldots k_2 - 1 \\ \frac{\cosh^\delta R_l}{\cosh^\delta t} & p = (t, \theta) \in \hat{C}_{R_l} \end{cases}$$

Let p_l be a point where the supremum is achieved. Notice we always have $|u_l(p)| \leq \rho_l(p)$, with equality at p_l. We will obtain various contradictions depending on where p_l occurs.

First consider the case where $p_l = (t_l, \theta_l) \in \hat{C}_{R_l}$ with $|t_l|$ bounded. In this case we restrict to $\hat{C}_{R_l}$ and renormalize by setting

$$\tilde{u}_l(t, \theta) = (\cosh^{-\delta} R_l)u_l(t, \theta).$$

Then choose a subsequence which converges uniformly on compact sets and such that $(t_l, \theta_l) \to (\bar{t}, \bar{\theta})$. In the limit we obtain a Jacobi field $\bar{u}$ for the Delaunay metric g_ϵ such that

$$|\bar{u}(t, \theta)| \leq \cosh^{-\delta} t$$

with equality at $(\bar{t}, \bar{\theta})$, which is a contradiction.

Next consider the case where $p_l = (t_l, \theta_l) \in C_{R_l}$ with $|t_l|$ and $|t_l \pm R_l|$ all unbounded. We will treat the instance where $t_l < 0$; the case where $t_l > 0$ is similar. In this case we restrict to the part of $\hat{C}_{R_L}$ parameterized by $(t, \theta) \in [-R_l - t_l, |t_l|] \times S^{n-1}$ and renormalize by setting

$$\tilde{u}_l(t, \theta) = \left(\frac{\cosh^\delta t_l}{\cosh^\delta R_l}\right) u_l(t + t_l, \theta).$$

With this renormalization $|\tilde{u}_l(0, \theta_l)| = 1$ and

$$\begin{aligned} |u_l(t, \theta)| &\leq \frac{\cosh^\delta t_l}{\cosh^\delta(t + t_l)} \\ &\leq 2^\delta(e^t + e^{-t-2t_l})^{-\delta} \leq 2^\delta e^{\delta t}. \end{aligned}$$

Extracting a convergent subsequence we obtain a Jacobi field $\bar{u}$ for g_ϵ such that $|\bar{u}(0, \bar{\theta})| = 1$ and $|\bar{u}(t, \theta)| \leq 2^\delta e^{\delta t}$, which is a contradiction.

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Next consider the case where $p_l = (t_{j,l}, \theta_{j,l}) \in B_{r_0}(p_j) \backslash \{p_j\}$ (for $j = 1 \dots k_1 - 1$) and $t_l \to \infty$. In this case we restrict to $B_{r_0}(p_j) \backslash \{p_j\}$, renormalize by setting

$$\tilde{u}_l(t_j, \theta_j) = e^{\delta t_{j,l}} u_l(t_j + t_{j,l}, \theta_j)$$

and argue as in the previous case. The case where $p_l = (\tau_{j,l}, \theta_l) \in B_{r_0}(q_j) \backslash \{q_j\}$ with $\tau_{j,l} \to \infty$ is similar.

Next consider the case where $p_l \in \Omega_1$, where Ω_1 is some fixed compact set containing M_1^c. Notice ρ_l is bounded and bounded away from 0 in Ω_1. Restrict to

$$M_1^c \cup (\cup_1^{k_1-1} B_{r_0}(p_j) \backslash \{p_j\}) \cup \{(t, \theta) \in \hat{C}_{R_l} : t < 0\}$$

and take a subsequence which converges uniformly on compact sets (and so $p_l \to \bar{p} \in \Omega_1$). Then in the limit we obtain a Jacobi field $\bar{u}$ on M_1 which decays exponentially near all p_j except p_0, and which has subexponential growth near p_0. Also, by the normalization $|u(\bar{p})| \neq 0$. By lemma 7, $\bar{u}$ must also decay like $e^{-\delta t_0}$ near p_0 for some $\delta > 1$, which contradicts the unmarked nondegeneracy of (M_1, g_1).

Finally, consider the case where $p_l \in \Omega_2$, where Ω_2 is a fixed compact set containing M_2^c. If we restrict to

$$M_2^c \cup (\cup_1^{k_2-1} B_{r_0}(q_j) \backslash \{q_j\}) \cup \{(t, \theta) \in \hat{C}_{R_l} : t > 0\}$$

and choose a convergent subsequence as we did in the previous case, we can only conclude that the limit $\bar{u}$ is asymptotic to $a v_\epsilon^{0,+}$ near q_0. At this point we rescale so that $a = 1$. Fix some $R_0 > 0$. Then there is an l_0 depending on R_0 such that for $l \geq l_0$

$$\|u_l - v_\epsilon^{0,+}\|_{C^{2,\alpha}([R_l - R_0 - 1, R_l] \times S^{n-1})} = O(e^{-\delta R_0}).$$

Moreover, if we restrict u_l to

$$M_1^c \cup (\cup_1^{k_1-1} B_{r_0}(p_j) \backslash \{p_j\}) \cup \{(t, \theta) \in \hat{C}_{R_l} : t < 0\}$$

we know by the previous argument that it must converge uniformly to zero. Thus for $l \geq l_0$

$$\|u_l\|_{C^{2,\alpha}([-R_l, -R_l + R_0 + 1] \times S^{n-1})} = O(e^{-\delta R_0}).$$

Recall that we can write the metric $g_l = g_{R_l, \phi}$ on $\hat{C}_{R_l}$ as

$$(u_\epsilon + v_l)^{\frac{4}{n-2}}(dt^2 + d\theta^2)$$

where

$$|v_l(t, \theta)| = O\left(\frac{\cosh^{\gamma_{n+1}(\epsilon)} t}{\cosh^{\gamma_{n+1}(\epsilon)} R_l}\right).$$

Thus $L_{g_l} - L_{g_\epsilon}$, when restricted to $\hat{C}_{R_l}$ a second order differential operator whose coefficients are $O(\frac{\cosh^\mu t}{\cosh^\mu R_l})$ on $\hat{C}_{R_l}$ for some $\mu \in (\delta, \gamma_{n+1}(\epsilon))$, which implies

$$L_{g_\epsilon}(u_l)(t, \theta) = O\left(\frac{\cosh^{\mu-\delta} t}{\cosh^{\mu-\delta} R_l}\right)$$

on $\hat{C}_{R_l}$. In other words, $L_{g_\epsilon}(u_l) \in C_{\delta-\mu}^{0,\alpha}([-R_l + R_0, R_l - R_0] \times S^{n-1})$ and

$$\|L_{g_\epsilon}(u_l)\|_{C_{\delta-\mu}^{0,\alpha}([-R_l + R_0, R_l - R_0] \times S^{n-1})} = O(e^{(\delta-\mu)R_0}).$$

Let

$$\tilde{u}_l = H_{R_l - R_0, \epsilon}(L_{g_\epsilon}(u_l)).$$

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Then

$$
\begin{aligned}
0 &= \int_{[-R_l+R_0, R_l-R_0]\times S^{n-1}} \tilde{u}_l L_{g_\epsilon}(v_\epsilon^{0,-}) - v_\epsilon^{0,-} L_{g_\epsilon}(\tilde{u}_l) \\
&= \int_{\{R_l-R_0\}\times S^{n-1}} (\tilde{u}_l \frac{\partial v_\epsilon^{0,-}}{\partial \nu} - v_\epsilon^{0,-}\frac{\partial \tilde{u}_l}{\partial \nu}) - \int_{\{-R_l+R_0\}\times S^{n-1}} (\tilde{u}_l \frac{\partial v_\epsilon^{0,-}}{\partial \nu} - v_\epsilon^{0,-}\frac{\partial \tilde{u}_l}{\partial \nu}) \\
&= \int_{\{R_l-R_0\}\times S^{n-1}} (v_\epsilon^{0,+}\frac{\partial v_\epsilon^{0,-}}{\partial t} - v_\epsilon^{0,-}\frac{\partial v_\epsilon^{0,+}}{\partial t} + O(e^{-\delta R_0} + e^{(\delta-\mu)R_0})) + \int_{\{-R_l+R_0\}\times S^{n-1}} O(e^{-\delta R_0}) \\
&= 1 + O(e^{-\delta R_0} + e^{(\delta-\mu)R_0}).
\end{aligned}
$$

This completes the proof of theorem 1.

∎

8   Questions

In this final section we raise some interesting questions related to this construction.

The first question is: how much of this gluing construction can be extended to arbitrary (complete, connected and noncompact) manifolds with constant positive scalar curvature? The first requirement we see is that all the ends must be asymptotically Delaunay. So it might be natural to apply this theorem with $M_i = \bar{M}_i \backslash \Lambda$ where $\bar{M}_i$ is a closed locally conformally flat manifold and Λ is a finite set. In this case the ends of M_i correspond to punctured neighborhoods of $p \in \Lambda$ and the metric is indeed asymptotically Delaunay there. However, we lack the conformal Killing fields which give rise to the asymptotic translations in this case (as we used in remark 2). However, one might be able to prove a similar result where one supposes that the localization of B_{g_1} to E_o is trivial, i.e. that there are no bounded Jacobi fields for g_1 which decay on all ends but E_0. This may not be such a strong hypothesis, because as argument similar to that of lemma 7 shows that the localization of B_{g_1} to E_0 can be at most 1-dimensional.

The second question stems from conversations with N. Korevaar and is: how much can we say about the global structure of the moduli space of complete scalar curvature metrics $n(n-1)$ metrics on $S^n \backslash \{p_1 \ldots p_k\}$? The simplest nontrivial case seems to be $k = 3$. We can conclude (using an Alexandrov reflection argument as in [CGS] and [KKS]) that these metrics must be symmetric under reflection through some equatorial S^{n-1} (after composing with a conformal motion of S^n). More precisely, one can use stereographic projection to turn the problem into a scalar PDE on $\mathbb{R}^n$ and take inversion through $n-1$-dimensional spheres centered at the origin in place of reflection through some hyperplane. The same arguments as in [KKS] hold. However, the main tool we lack in the scalar curvature case is a way to get necksize bounds on the ends, and reconstruct the metric from the asymptotic necksizes as in [GKS]. Grosse-Brauckmann, Kusner and Sullivan use a conjugate minimal surface in S^2 to classify all three ended, genus zero constant mean curvature surfaces in [GKS]. At the present, we do not have any way to either find necksize bounds or show that all possible combination of necksizes allowed by balancing is realized.

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