# Fall 2003 REU Report: Basic Jet Scheme Computation

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H. Nielsen Report REU Fall 2003

The first thing I did in this research project was skim through actual papers, given to me by Professor Enescu, that treated the topic of jet scheme computation on which the project was based. Although I found it very difficult to follow much of the complicated math that was involved, I was exposed to the main ideas and general concepts thus orienting me to the source of the subject matter. After going through this reading material I started building on my novice familiarity with abstract algebra to develop a sufficient background in commutative algebra and algebraic geometry in order to understand the concepts involved in the computation of jet schemes. Professor Enescu helped me find appropriate text books that treated the subject at a pace suited for my level of understanding. The two main books I used were Commutative Algebra for Undergraduates by Miles Reid and Introduction to Commutative Algebra by Atiyah and MacDonald.

Twice a week Professor Enescu and I met for two hours and each time I was given a reading assignment and exercises pertaining to it from the above-mentioned books to have ready for the following meeting. The main concepts that were concentrated on were the fundamental concepts of prime ideals, maximal ideals, radicals, nilpotent elements, irreducibility, etc. The study of these concepts was aimed at understanding the Zariski topology and also how to interpret varieties, or schemes, in terms associated with ring theory. This was essential because the research project refers to the notion of the irreducibility of varieties which is closely related to the concept of prime ideals.

As I worked with this material in the text books I gained the mathematical tools necessary for working with the notion of a jet scheme associated with a hypersurface which will be defined and explored in continuation. A variety (or a scheme) is given by a set of polynomial equations $f_1, \dots, f_m \in [x_1, \dots, x_n]$ by looking at the set of common zeros $Z(f_1, \dots, f_m) = \{(x_1, \dots, x_n) \mid f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0\} \subseteq C^n$. For example $Z(x^2-y^2) = \{(x,y) \mid x^2 = y^2\} = \{(x,y) \mid x = \pm y\}$, which is the union of two lines: $y = -x$ and $y = x$ (as a subset of $C^2$).

These varieties (or schemes) can be studied using ring theory. To each variety $Z(f_1, \dots, f_m)$, one can associate the coordinating ring of functions $C[x_1, \dots, x_n]/(f_1, \dots, f_m)$, where $(f_1, \dots, f_m)$ is the ideal generated by the polynomials $f_1, \dots, f_m$. This allows the in-depth understanding of the properties of a given variety. For example, for $f = x$, $V(x) = \{0\} \subseteq C$. In this example the ring of functions is $C[x]/(x)$ which is isomorphic to C which is a field. But now consider the equation $f = x^2$. $V(x^2) = \{0\} \subseteq C$. This time the ring of functions is $C[x]/(x^2)$ which is not a field. Each ring $R = C[x_1, \dots, x_n]/(f_1, \dots, f_m)$ has a spectrum Spec(R). On Spec(R) one has the Zariski topology where the closed sets are $V(I) = \{I \subseteq p \mid p \text{ is prime}\}$ and I is an ideal in R. This topology has a counterpart on $Z(f_1, \dots, f_m) \subseteq C^n$ where the closed sets are of the form Z(I) and $I \subseteq C[x_1, \dots, x_n]/(f_1, \dots, f_m)$. For example take $R = C[x,y]/(x^2-y^3)$. $Z(x^2-y^3) = \{(x,y) \mid x^2 = y^3\} \subseteq C^2$. The closed sets on $\{(x,y) \mid x^2 = y^3\}$ are sets of the form Z(I) with $I \subseteq C[x,y]/(x2 - y3)$. For example, $I = (x,y)/(x^2 - y^3)$ will give a closed set, namely $Z(x,y) = \{(0,0)\}$.

As an additional example take $R = C[x,y]$. In this case $C^2$ is a variety. A couple examples of closed sets in $C^2$ are the following: Z(I) and $I_i \subseteq C[x,y]$ where $I_1 = (x - y)$, $I_2 = (x - y^2)$, $I_3 = (x^2, xy)$, $I_4 = (x^2 - y^2 - 1)$. The graphs of the polynomial expressions of these ideals (e.g. lines, parabolas, hyperbolas, etc.) are examples of the basic building blocks of algebraic geometry. Studying these closed sets is the same as studying the Zariski topology on the spectrum of a ring. In fact, there exists a one-to-one correspondence between them. This one-to-one correspondence reverses inclusion. For example, usually maximal ideals correspond to the largest subset that is not contained in any subset except for the whole set. But in this case maximal ideals correspond to individual points – the smallest things in the set. And similarly prime ideals, which are usually thought of as smaller subsets than maximal ideals, correspond to objects larger than the points to which maximal ideals correspond in this peculiar point of view.

The principle focus of this research project was to find a method of computation for the jet schemes of a given polynomial (with isolated singularities), without having to going through the lengthy step-by-step computation which will now be outlined. Let $f = f(x_{0,1}, \ldots, x_{0,n}) = 0$ be a hypersurface in $C^n$. For $f(x_{0,1},x_{0,2}) = x_{0,1} - (x_{0,2})^2$, $Df = \partial f/\partial x_{0,1} \cdot x_{1,1} + \partial f/\partial x_{0,2} \cdot x_{1,2} = \partial f/\partial x_{0,1} \cdot Dx_{0,1} + \partial f/\partial x_{0,2} \cdot Dx_{0,2} = 1 \cdot x_{1,1} - 2 \cdot x_{0,2} \cdot x_{1,2}$ where $D(x_{i,j}) = x_{i+1,j}$ . D is an operator that behaves like a derivation i.e. it obeys the product rule. For example $D(x_{2,1} \cdot x_{1,1}) = D(x_{2,1}) \cdot x_{1,1} + x_{2,1} \cdot D(x_{1,1}) = x_{1,1} \cdot x_{1,1} + x_{2,1} \cdot x_{2,1}$.

Definition: If $f = f(x_{0,1}, \ldots, x_{0,n})$, then

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{0,i}} \cdot D(x_{0,i}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{0,i}} \cdot x_{1,i}$$

Inductively $D^2(f) = D(D(f))$ and $D^{m+1}(f) = D(D^m(f))$:

$$D^2(f) = D\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_{0,i}} \cdot x_{1,i}\right) = \sum_{i=1}^{n} D\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_{0,i}} \cdot x_{1,i}\right) = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_{0,j} \cdot \partial x_{0,i}} \cdot x_{1,j} \cdot x_{1,i} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_{0,i}} \cdot x_{2,i}$$

The first three jets are as follows:

$0^{th}$ jet: $X = Z(f) = \{f(x_{0,1}, \ldots, x_{0,n}) = 0\} \subseteq C^n$ ; $(x_{0,1}, \ldots, x_{0,n})$

$1^{st}$ jet: $X_1 = \{f(x_{0,1}, \ldots, x_{0,n}) = 0; D(f) = 0 \Leftrightarrow \sum_{i=1}^{n} \partial f/\partial x_{0,i} \cdot x_{1,i} = 0\} \subseteq C^n \times C^n$

$(x_{0,1}, \ldots, x_{0,n}, \; x_{1,1}, \ldots, x_{1,n})$

2nd jet: $X_2 = \{f = 0; D(f) = 0; D^2(f) = 0\} \subseteq C^n \times C^n \times C^n$ (See previous page for $D^2(f)$)

$(x_{0,1}, \ldots, x_{0,n}, \; x_{1,1}, \ldots, x_{1,n}, \; x_{2,1}, \ldots, x_{2,n})$

Thus the 0th jet, $Z(f) = 0$, corresponds to the original hypersurface and the 1st jet corresponds to the tangent space $\{(x,y) \mid x \text{ in } Z(f); y \text{ in tangent plane at } x\}$ at the hypersurface, etc.

To arrive at the goal of a direct computation for the $n^{th}$ jet it is noted that f has rational singularities if for all m the jet schemes $x_m$ are irreducible. Remember that a topological space is irreducible if it cannot be written as a nontrivial union of two closed sets. Paraphrasing Mustata's Theorem, if an object is not made of one piece, then the hypersurface f will have "bad" singularities. So the goal of this project boils down to the question of when $X_m$ is irreducible. The method employed to accomplish this is looking at the ideal $(f, D(f), D^2(f), \ldots, D^m(f)) = P_m$. Now $Z(P_m) = X_m$. $X_m$ is irreducible $\Leftrightarrow$ $P_m$ is prime. Hence another important question: When is $P_m$ prime?

The plan involves understanding these equations $f, D(f), \ldots, D^m(f)$ so that it can be established whether $P_m$ is prime or not. To obtain a general description of the jet equations $f, D(f), D^2(f), \ldots$ the first step is to figure out the one variable case $f(x_1^0)$. In this case the jet equation can be computed directly from the coefficients of the original polynomial. It follows a recurrence relation as follows:

First it should be noted that while the polynomial representing the hypersurface can have any number of variables in it, here only a polynomial with one variable will be treated and those polynomials with more than one variable will follow similarly. So let f(x) be an arbitrary polynomial in a single variable. Note that $D(x_i) = x_{i+1}$. When more than one variable is used, the different variables are distinguished by a varying subscript, i.e. $f(x_1, x_2, \ldots, x_n)$ or $f(x_{0,1}, \ldots, x_{0,n})$ where in the calculation $\partial f/\partial x_{0,1} = x_{1,1}$ and $\partial f/\partial x_{1,1} = x_{2,1}$ and so forth since $D(x_{i,j}) = x_{i+1,j}$ as mentioned above. In this one variable case the subscript will be omitted. If an exponent were to be used then the variable would be surrounded by parentheses. The first through the sixth jet schemes are listed here:

$D'$: $f' \cdot x_1$

$D^2$: $f' \cdot x_2 + f^{(2)} \cdot x_1 \cdot x_1$

$D^3$: $f' \cdot x_3 + f^{(2)} \cdot (2x_1x_2 + x_2x_1) + f^{(3)} \cdot (x_1x_1x_1)$

$= f' \cdot x_3 + 3\, f^{(2)} \cdot (x_2x_1) + f^{(3)} \cdot (x_1)^3$

$D^4$: $f' \cdot x_4 + f^{(2)} \cdot (3x_1x_3 + 3x_2x_2 + x_3x_1) + f^{(3)} \cdot (3x_1x_1x_2 + 2x_1x_2x_1 + x_2x_1x_1) +$

$f^{(4)} \cdot (x_1x_1x_1x_1)$

$= f' \cdot x_4 + f^{(2)} \cdot (4x_1x_3 + 3x_2x_2) + f^{(3)} \cdot (6x_1x_1x_2) + f^{(4)} \cdot (x_1)^4$

$D^5$: $f' \cdot x_5 + f^{(2)} \cdot (4x_1x_4 + 6x_2x_3 + 4x_3x_2 + x_4x_1) + f^{(3)} \cdot (6x_1x_1x_3 + 8x_1x_2x_2 + 3x_1x_3x_1$

$+4x_2x_1x_2 + 3x_2x_2x_1 + x_3x_1x_1) + f^{(4)} \cdot (4x_1x_1x_1x_2 + 3x_1x_1x_2x_1 + 2x_1x_2x_1x_1$

$+x_2x_1x_1x_1) + f^{(5)} \cdot (x_1x_1x_1x_1x_1)$

$= f' \cdot x_5 + f^{(2)} \cdot (5x_1x_4 + 10x_2x_3) + f^{(3)} \cdot (10x_1x_1x_3 + 15x_1x_2x_2) + f^{(4)} \cdot (10x_1x_1x_1x_2)$

$+ f^{(5)} \cdot (x_1)^5$

$D^6$: $f' \cdot x_6 + f^{(2)} \cdot (5x_1x_5 + 10x_2x_4 + 10x_3x_3 + 5x_4x_2 + x_5x_1) + f^{(3)} \cdot (10x_1x_1x_4 +$

$20x_1x_2x_3 + 15x_1x_3x_2 + 4x_1x_4x_1 + 10x_2x_1x_3 + 15x_2x_2x_2 + 6x_2x_3x_1 + 5x_3x_1x_2$

$+ 4x_3x_2x_1 + x_4x_1x_1) + f^{(4)} \cdot (10x_1x_1x_1x_3 + 15x_1x_1x_2x_2 + 6x_1x_1x_3x_1 +$

$10x_1x_2x_1x_2 + 8x_1x_2x_2x_1 + 3x_1x_3x_1x_1 + 5x_2x_1x_1x_2 + 4x_2x_1x_2x_1 + 3x_2x_2x_1x_1$

$+ x_3x_1x_1x_1) + f^{(5)} \cdot (5x_1x_1x_1x_1x_2 + 4x_1x_1x_1x_2x_1 + 3x_1x_1x_2x_1x_1 +$

$2x_1x_2x_1x_1x_1 + x_2x_1x_1x_1x_1) + f^{(6)} \cdot (x_1x_1x_1x_1x_1x_1)$

$= f' \cdot x_6 + f^{(2)} \cdot (6x_1x_5 + 15x_2x_4 + 10x_3x_3) + f^{(3)} \cdot (15x_1x_1x_4 + 60x_1x_2x_3 + 15x_2x_2x_2) + f^{(4)} \cdot$

$(20x_1x_1x_1x_3 + 45x_1x_1x_2x_2) + f^{(5)} \cdot (15x_1x_1x_1x_1x_2) + f^{(6)} \cdot (x_1)^6$

We want to be able to predict the coefficients of the jet schemes. Definition:

$$D^m = \sum_{n=1}^{m} C_m^n$$

$$C_m^n = \sum_A \left(x_{i_1}, \dots, x_{i_n} \cdot b^m_{i_1, \dots, b_{i_n}}\right) \text{ where } A = \{(i_1, \dots, i_n) \mid \sum i_j = m\}$$

Theorem (Recurrence formula):

$$b^m_{i_1, \dots, i_n} = \sum_{j=1}^{n} \left(b_{i_1, \dots, i_j - 1, \dots, i_n}\right)^{(m-1)}$$

- $b_1^1 = 1$
- $b^K_{0, i_2, \dots, i_n} = b^K_{i_2, \dots, i_n}$
- $b^1_{i_1, \dots, i_j, \dots, i_n} = 0$ if $i_j = 0$ for some $j \geq 2$

Proposition:

1) $b_n^n = 1$
2) $b^n_{\underbrace{1,1,\dots,1}_{n}} = 1$

Proof:

1) $b_n^n = b_{n-1}^{n-1} = \dots = b_1^1 = 1$
2) $b^n_{1,1,\dots,1} = b^{n-1}_{0,1,\dots,1} + \cancel{b^{n-1}_{1,0,1,\dots,1}}^{0} + \dots + \cancel{b^{n-1}_{1,1,\dots,0}}^{0}$

$$= b^{n-1}_{1,1,\dots,1} = \dots = b_1^1 = 1$$

Proposition:

$b^n_{\ell,n-\ell}$ is a polynomial in $n$ of degree $\ell$ for $n >> 0$.

Proof:

By induction on $\ell$.

$\ell=1$: $b^n_{1,n-1} = b^{n-1}_{n-1} + b^{n-1}_{1,n-2}$

$= 1 + b^{n-1}_{1,n-2}$

Call $x_n = b^n_{1,n-1}$ $\qquad b^2_{1,1} = x_2 = 1$

$$\begin{aligned} x_n &= 1 + \cancel{x_{n-1}} \\ \cancel{x_{n-1}} &= 1 + \cancel{x_{n-2}} \\ &\vdots \\ + \quad \cancel{x_3} &= 1 + x_2 \\ \hline x_n &= (n-2) + x_2 \\ &= (n-2)+1 \\ &= \underline{\underline{n-1}} \checkmark \end{aligned}$$

$\ell=2$: $b^n_{2,n-2} = b^{n-1}_{1,n-2} + b^{n-1}_{2,n-3}$

$b^{n-1}_{1,n-2} = x_{n-1} = n-2$

Call $y_n = b^n_{2,n-2}$

$y_n = n-2 + b^{n-1}_{2,n-3}$

$y_n = (n-2) + y_{n-1}$

$b^2_{2,0} = 0$

$b^3_{2,1} = b^2_{2,0} + b^2_{1,1} = 1 = y_3$

$$
\begin{aligned}
y_n &= (n-2) + \cancel{y_{n-1}} \\
\cancel{y_{n-1}} &= (n-3) + \cancel{y_{n-2}} \\
&\ \vdots \\
+ \quad \cancel{y_4} &= 2 + y_3
\end{aligned}
$$

$$y_n = (2 + \cdots + n-2) + y_3$$

$$y_n = (1 + \cdots + n-2) - 1 + y_3$$

$$y_n = (1 + \cdots + n-2) - 1 + 1$$

$$y_n = \frac{(n-2)(n-1)}{2}, \text{ since } \sum_{i=1}^{K} i = \frac{K(K+1)}{2}$$

$$b^n_{2,n-2} = \frac{(n-2)(n-1)}{2} \checkmark$$

$l=3$: $b^n_{3,n-3} = b^{n-1}_{2,n-3} + b^{n-1}_{3,n-4}$

Call $z_n = b^n_{3,n-3}$

$$z_n = \frac{(n-3)(n-2)}{2} + z_{n-1}$$

$$
\begin{aligned}
z_n &= \frac{n^2-5n+6}{2} + \cancel{z_{n-1}} \\
\cancel{z_{n-1}} &= \frac{(n-1)^2-5(n-1)+6}{2} + \cancel{z_{n-2}} \\
&\ \vdots \\
+ \quad \cancel{z_5} &= 3 + z_4 \qquad \left(z_4 = b^4_{3,1} = 1\right)
\end{aligned}
$$

$$z_n = \sum_{k=5}^{n} \frac{k^2-5k+6}{2} + 1$$

$\Longrightarrow$ $b^n_{3,n-3}$ is a polynomial in $n$ of degree 3 $\checkmark$

$$b^n_{\ell,n-\ell} = b^{n-1}_{\ell-1,n-\ell} + b^{n-1}_{\ell,n-\ell-1}$$

Assume $b^{n-1}_{\ell-1,n-\ell}$ a polynomial of degree $\ell-1$ in $n-1 = P_{\ell-1}(n-1)$

call $b^n_{\ell,n-\ell} = w_n$.

$$w_n = P_{\ell-1}(n) + \cancel{w_{n-1}}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$+ \quad \cancel{w_{\ell+2}} = P_{\ell-1}(\ell) + w_{\ell+1}$$

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$$w_n = \sum_{K=\ell}^{n} P_{\ell-1}(K) + w_{\ell+1}$$

$$P_{\ell-1}(K) = a_{\ell-1}K^{\ell-1} + \dots$$

$$w_n = a_{\ell-1}\sum_{K=\ell}^{n}(K^{\ell-1}) + \dots$$

$\sum_{K=\ell}^{n}(K^{\ell-1})$ represents a polynomial of degree $\ell$.

$$w_n = a_{\ell-1}\cdot\frac{K^\ell}{\ell} + \dots \quad \checkmark$$

Proposition:

$b^n_{\ell,n-\ell}$ is a polynomial in $n$ of degree $\ell$ and leading coefficient $\frac{1}{\ell!}$ for $n \gg 0$. The polynomial is independent of $n$.

Proof: By induction on $\ell$.

$\ell=1$: $P_1(n) = n-1$. Leading coefficient $= \frac{1}{1!} = 1$ ✓

$\ell=2$: $P_2(n) = \frac{(n-2)(n-1)}{2}$. Leading coefficient $= \frac{1}{2!} = \frac{1}{2}$,

$$b^n_{\ell,n-\ell} = b^{n-1}_{\ell-1,n-\ell} + b^{n-1}_{\ell,n-\ell-1}$$

$$b^n_{\ell,n-1} = \omega_n$$

$$b^{\ell+1}_{\ell,1} = \cancel{b^{\ell}_{\ell,0}}^{\,0} + b^{\ell}_{\ell-1,1} = P_{\ell-1}(\ell)$$

$$\omega_n = P_{\ell-1}(n-1) + \cancel{\omega_{n-1}}$$

$$\vdots$$

$$+\quad \cancel{\omega_{\ell+2}} = P_{\ell-1}(\ell+1) + \omega_{\ell+1}$$

$$\omega_n = \sum_{k=\ell+1}^{n-1} P_{\ell-1}(k) + \omega_{\ell+1}$$

$$\omega_n = \sum_{k=\ell}^{n-1} P_{\ell-1}(k)$$

$$\omega_n = \sum_{k=\ell}^{n-1} \left( \frac{1}{(\ell-1)!} k^{\ell-1} + \cdots \right)$$

$$\omega_n = \sum_{k=1}^{n-1} \left( \frac{1}{(\ell-1)!} k^{\ell-1} + \cdots \right) - C \text{ (for some } C\text{)}$$

Using Bernoulli property again...

$$\omega_n = \frac{1}{(\ell-1)!} \cdot \frac{1}{\ell} \cdot k^{\ell} + \cdots$$

$$\Rightarrow \omega_n = \frac{1}{\ell!} \cdot k^{\ell} + \cdots$$

$$\Rightarrow b^n_{\ell,n-\ell} = \frac{1}{\ell!} \cdot k^{\ell} + \cdots \quad \checkmark$$