Nika Polevaya
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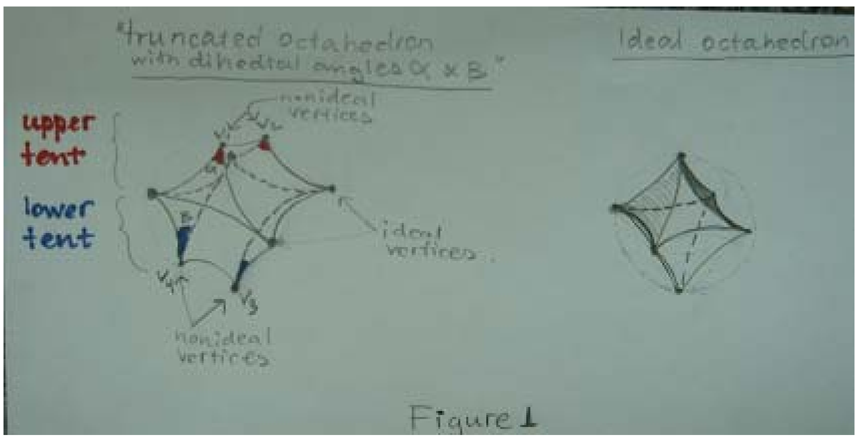
TRUNCATED HYPERBOLIC OCTAHEDRON AND ITS GROUP OF ISOMETRIES

The hyperbolic object that will be under consideration is a truncated hyperbolic octahedron (herein “Octy”). Four of the vertices of Octy that lie in a plane are ideal, i.e. reside on the boundary of the Hyperbolic space. The other two (axial) vertices are “cut off” and turned into edges bounded by the dihedral angles α in the upper hemisphere and β in the lower hemisphere. Octy thus has 8 total vertices – 4 ideal and 4 nonideal. Please refer to **Figure 1**. (*)It is important to notice that all of the hyperbolic sphere intersections are at 90 degrees, except for the 2 dihedral angles mentioned above.



My **GOAL** was to determine the group of Mobius transformations that would map the nonintersecting sides of Octy to each other, and to express these transformations in terms of α, β, and the scaling parameter R.

Step 1: Construct Octy explicitly.

The outline of the strategy is quite simple: one can reduce the 3D problem to a 2 D problem by consider the boundaries of 8 hyperbolic spheres that harbor the sides of Octy. Construct the upper part of Octy (the “tent” with dihedral angle α (**Figure2**).

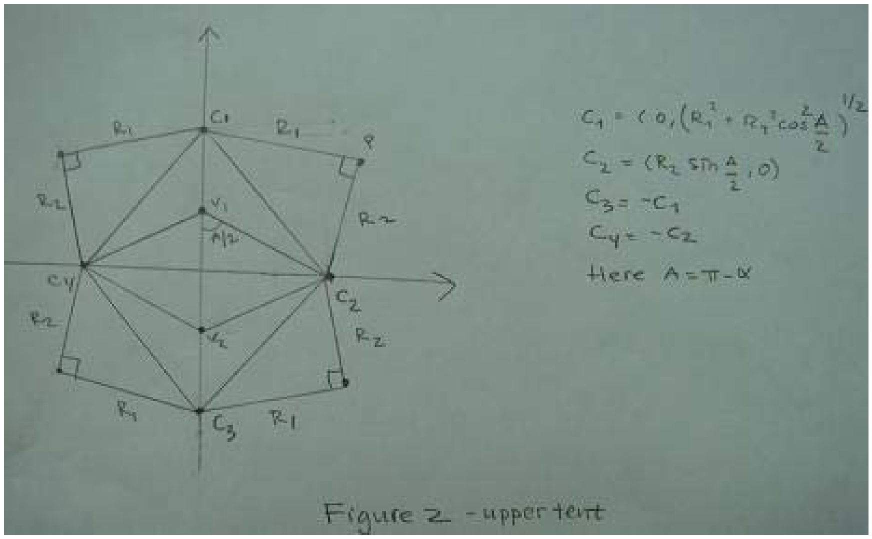


Figure 2 - upper tent

Then construct the lower part of Octy with dihedral angle β (**Figure3**), which will be the same as for the upper "tent" but dependent on β instead, and rotated by 90 degrees.

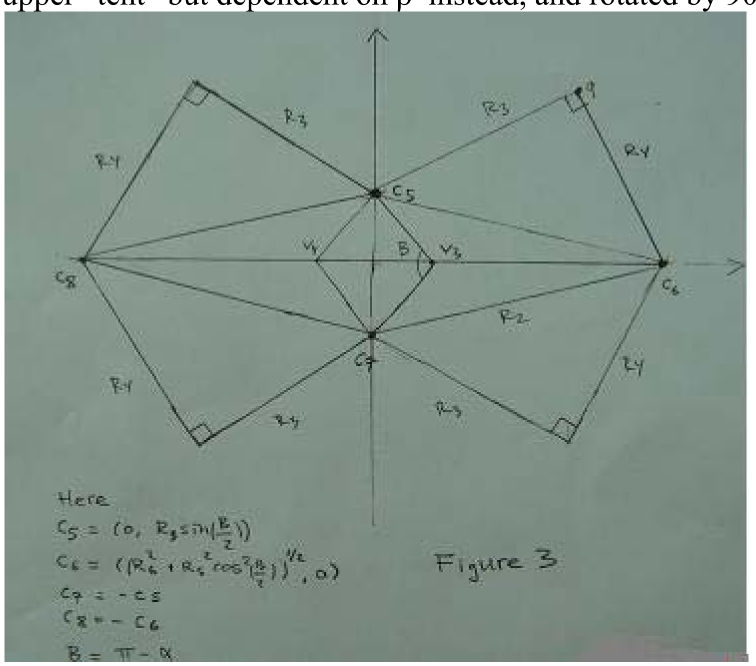
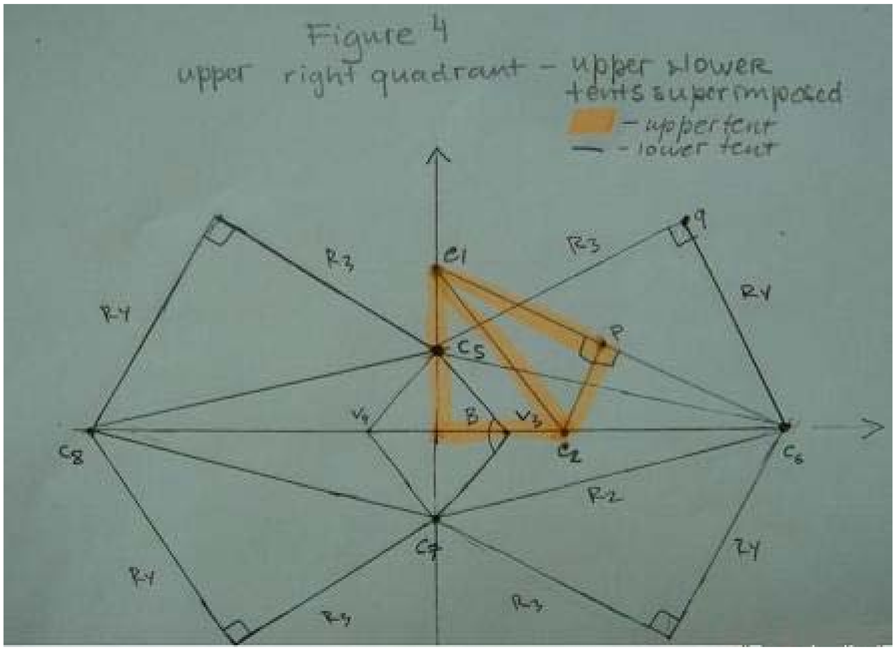


Figure 3

Then fit the two "tents" together in a way that the condition (*) is satisfied (**Figure4**).



Note: on the figures here, c1....c8 are the centres of the hyperbolic spheres that harbor the faces of Octy, and R1....R8 are their radii. Since R(n)=R(n+2), $1<n<8$, we can just, consider R1, R2, R5, R6.

Notice the symmetry with respect to y and x axes. Thus one can consider only the upper right quadrant. Notice also that c1.....c8 can be expressed in terms of R1, R2, R5, R6 and α and β:

```
c1 := (R1^2+R2^2*cos(1/2*A)^2)^(1/2)
c2 := R2*sin(1/2*A)
c5 := R5*sin(1/2*B)
c6 := (R6^2+R5^2*cos(1/2*B)^2)^(1/2)
(here A=π-α, B=π-β)
```

Splicing the "tents" together and condition (*) provide us with **3 nondegenerate equations** and 4 unknowns:

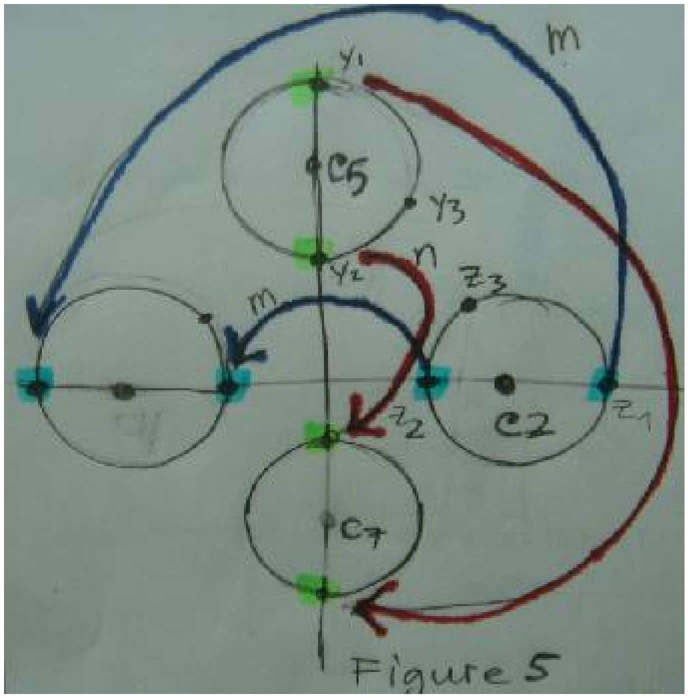
```
eqn1 := R2^2+R6^2 = ((R6^2+R5^2*cos(1/2*B)^2)^(1/2)-R2*sin(1/2*A))^2
eqn2 := R2^2*cos(1/2*A)^2+R5^2*cos(1/2*B)^2 = 2*R1*R6
eqn3 := R1^2+R5^2 = ((R1^2+R2^2*cos(1/2*A)^2)^(1/2)+R5*sin(1/2*B))^2
```

Thus we can express R1, R2, R5, R6 (hence c1....c8) in terms of R6 and α and β, as desired:

```
r1 := sin(1/2*A)^2*R6*(1+(1-
cos(1/2*A)^2*cos(1/2*B)^2)^(1/2))/(cos(1/2*A)^2*sin(1/2*B)^2)
r2 := (1+(1-
cos(1/2*A)^2*cos(1/2*B)^2)^(1/2))*R6*sin(1/2*A)/(cos(1/2*A)^2*sin(1/2*B))
r5 := R6*sin(1/2*A)/sin(1/2*B)
```

Step 2: Constructing Mobius transformations.

Consider the following cross-section of Octy at the boundary at infinity (**Figure 5**).



Consider three points on C2 (which is a subset of S2, which harbors the 2nd face of Octy), and their image under the map m which will map side 2 to side 4. Here z1->-z1, z2->-z2, z3-> -Re(z3)+Im(z3). These 3 conditions completely determine m since m is a Mobius transformation. Since m can be expressed as m(z)=(az+b)/(cz+d), our task is to find a,b,c,d.

d = (-z3*z1-z3*z2+z1*z2+Re(z3)*z3-I*Im(z3)*z3)*c/(z3-Re(z3)+I*Im(z3))
a = -(-z1*Re(z3)+I*z1*Im(z3)-z2*Re(z3)+I*z2*Im(z3)+z1*z2+Re(z3)*z3-I*Im(z3)*z3)*c/(z3-Re(z3)+I*Im(z3))
b = z1*z2*c
c = c

Now remember that z1, z2, z3 can be expressed in terms of R6 and α and β. Hence, m can be uniquely expressed in terms of these parameters.

In a similar way we can obtain a Mobius transformation n that maps side 5 to side 7, and thus C5 to C7 by considering points y1, y2, y3 on C5 and their images on C7. The two Mobius transformations m & n generate the group of isometries of Octy. To check whether we have correctly constructed m & n, we can check that the commutator of m & n is parabolic. The commutator h is parabolic, iff it has only one fixed pt. Thus discriminant of the equation h(z)=z should be equal to 0, which is an easy check.

In summary, here we have explicitly constructed the truncated right-angled hyperbolic octahedron and its group of isometries in terms of the dihedral angles α and β, which reflect the extent of "truncation," and the scaling parameter R6.