Inverse Problems in Additive Number Theory

Abstract:

Additive number theory is the study of sums of sets, or sumsets. For example the sumset $A + B = \{a + b : a \in A, b \in B\}$. In inverse additive number theory problems information is know about the sumset and information about the original sets is deduced. One interesting problem to study is finding limits of sumsets; this is a direct problem. However, finding information about the sets which cause the extreme sumsets is an even more interesting inverse problem. This is what I will focus on.

Let all sets henceforth be strictly increasing sets of integers. Also let $A_0 + A_1 + ... + A_{h-1}$ be the sumset defined by $\{a_0 + a_1 + ... + a_{h-1} : a_i \in A_i\}$. If $A_i = A$ for $i \in [0,h-1]$ then the sumset $A_0 + A_1 + ... + A_{h-1}$ is written hA and is called the h-fold sumset of A.

Lets first look at the simple case of 2A. Let $A = \{ a_0 + a_1 + ... + a_{h-1}\}$.

Show that |2A| is minimal if and only if A is an arithmetic progression.

$$a_0 + a_0 < a_0 + a_1 < a_1 + a_1 < ... < a_{k-2} + a_{k-1} < a_{k-1} + a_{k-1} \quad (1)$$

This gives |2A| lower bound, namely

$$|2A| \geq 2(k - 1) + 1 = 2k - 1$$

This is the direct problem. Now that we know something about the sumset (the minimal size) we can try to attain information about A.

If $|2A| = 2k - 1$ then all elements of 2A are in the set (1) and can be written as $2a_i$ or $a_i + a_{i+1}$. Specifically,

$$2A = \{2a_i : i \in [0,k-1]\} \cup \{a_i + a_{i+1} : i \in [0,k-2]\}$$

Since

$$a_{i-1} + a_i < 2a_i < a_i + a_{i+1} \text{ and } a_{i-1} + a_i < a_{i-1} + a_{i+1} < a_i + a_{i+1}$$

then

$$2a_i = a_{i-1} + a_{i+1} \text{ or } a_i - a_{i-1} = a_{i+1} - a_i.$$

Thus $|2A| = 2k - 1$ if and only if A is an arithmetic progression.

And now the most generalized sumset $A_1 + A_2 + ... + A_h$ where $|A_i| = k(i)$. Let $a_{i,j}$ be the j^{th}

element of the A_i, with $j \in [0,k(i)-1]$.

$$a_{1,0} + a_{2,0} + \dots + a_{h,0} <$$
$$< a_{1,0} + a_{2,0} + \dots + a_{h,1} < \dots < a_{1,0} + a_{2,0} + \dots + a_{h,k(h)-1} < \dots$$
$$\vdots$$
$$\dots < a_{1,1} + a_{2,k(2)-1} + \dots + a_{h,k(h)-1} < \dots < a_{1,k(1)-1} + a_{2,k(2)-1} + \dots + a_{h,k(h)-1}$$

so

$$|A_1 + A_2 + \dots + A_h| \geq 1 + k(h) - 1 + \dots + k(1) - 1 = |A_1| + |A_2| + \dots + |A_h| - h + 1 \qquad (2)$$

Show that $|A_1 + A_2 + \dots + A_h|$ is minimal if and only if $A_1, \dots, A_h$ are each arithmetic progressions with the same common difference.

Part 1: *Show that if $A_1, \dots, A_h$ are each arithmetic progressions with the same common difference then $|A_1 + A_2 + \dots + A_h|$ is minimal.*

Let $A_i = a_{i,0} + d[0,k(i) - 1]$ for $i \in [1,h]$. Then

$$A_1 + A_2 + \dots + A_h = a_{1,0} + \dots + a_{h,0} + d[0 , k(1) + \dots + k(h) - h]$$
$$|A_1 + A_2 + \dots + A_h| = k(1) + \dots + k(h) - h + 1 = |A_1| + |A_2| + \dots + |A_h| - h + 1$$

Thus if $A_1, \dots, A_h$ are each arithmetic progressions with the same common difference then $|A_1 + A_2 + \dots + A_h|$ is minimal.

Part 2: *Show that if $|A_1 + A_2 + \dots + A_h|$ is minimal then $A_1, \dots, A_h$ are each arithmetic progressions with the same common difference.*

> Part 2 of this proof requires the assumption that for two sets of length m and n the minimal cardinality of the sumset, $m + n - 1$ (by (2)), occurs if and only if the two sets are arithmetic progressions with the same common difference. I have written up a horrendously inefficient proof of this fact, but given it's atrocity I will omit it for the sake of the reader. Suffice it to say that it is true. (3)

Assume that part 2 of the proof is true for h – 1. That is, assume

$$|A_1 + \ldots + A_{h-1}| = |A_1| + \ldots + |A_{h-1}| - h + 2$$

implies that

$$A_i = a_{i,0} + d[0,k(i) - 1] \text{ for } i \in [1,h-1] \qquad (4)$$

First, we know

$$|A_1 + \ldots + A_{h-1}| \geq |A_1| + \ldots + |A_{h-1}| - h + 2 \quad \text{(by (2))} \qquad (5)$$

Given:

$$|A_1 + \ldots + A_h| = |A_1| + \ldots + |A_h| - h + 1$$

then

$$\begin{aligned} |A_1| + \ldots + |A_h| - h + 1 &= |A_1 + \ldots + A_h| \quad . \\ &\geq |A_1 + \ldots + A_{h-1}| + |A_h| - 1 \quad \text{(by (2))} \qquad (6) \\ &\geq |A_1| + \ldots + |A_{h-1}| - h + 2 + |A_h| - 1 \quad \text{(by (5))} \\ &= |A_1| + \ldots + |A_h| - h + 1 \end{aligned}$$

it follows that

$$|A_1 + \ldots + A_{h-1}| + |A_h| - 1 = |A_1| + \ldots + |A_h| - h + 1$$

and thus

$$|A_1 + \ldots + A_{h-1}| = |A_1| + \ldots + |A_{h-1}| - h + 2$$

By (4)

$$A_i = a_{i,0} + d[0,k(i) - 1] \text{ for } i \in [1,h-1]$$

Repeating the process excluding A1 instead of Ah in step (6) will give

$$A_i = a_{i,0} + d[0,k(i) - 1] \text{ for } i \in [2,h]$$

So

$$A_i = a_{i,0} + d[0,k(i) - 1] \text{ for } i \in [1,h]$$

Therefore if part 2 of the proof is true for h – 1 sets then it will be true for h sets. Since we assume in (3) that part 2 is true for h = 2 and it is obviously true for the trivial case, h = 1, then *if* $|A_1 + A_2 + \ldots + A_h|$ *is minimal then* $A_1, \ldots, A_h$ *are each arithmetic progressions with the same common difference for all h.*

Thus $|A_1 + A_2 + \ldots + A_h|$ *is minimal if and only if* $A_1, \ldots, A_h$ *are each arithmetic progressions with the same common difference.*