M3210 Supplemental Notes: Basic Logic Concepts, by Anne D. Roberts

In this course we will examine statements about mathematical concepts and relationships between these concepts (definitions, theorems). We will also consider ways to determine whether certain statements are true or false (methods of proof). Our discussion uses everyday language that is often imprecise or ambiguous, yet the topics we plan to discuss are very precise and the methods of proof are based on the rules of logic. To avoid the problems that this disconnect between everyday language and mathematical discourse presents, we need to be specific about the terms we will use and what will be considered acceptable arguments. For this reason we will begin the course with a brief look at what is involved in 'mathematical discourse', its language and process of reasoning.

Section 1. Statements and Truth Tables

1.1 Simple Statements

According to the American College Dictionary, a statement is defined as; something stated; a declaration in speech or writing setting forth facts; an abstract of an account; act or manner of stating something. According to that definition, the following sentences are statements: The US calendar year begins in April. Oh, my goodness! Don't run. The house is ugly. In mathematics however the notion of a statement is more precise.

Definition 1.1: A mathematical statement is a declarative sentence that is true or false, but not both.

So, of the three sentences above, only the first one is a statement in the mathematical sense. Its truth value is false. 'Oh, my goodness!' is an exclamation, 'Don't run.' is a command and so neither of these sentences have truth value. In the last sentence, what is ugly may be a matter of disagreement so whether it is true or false is ambiguous and moreover which house is intended is not clear.

Some examples of mathematical statements are: five is less than eight; a positive rational number is the ratio of two natural numbers; $(2 + 4)^2 = 2^2 + 4^2$. Here the first two sentences are true and the third sentence is false.

While the expression, 3x + 2 = 10, is a mathematical sentence, it is not a mathematical statement since it involves the variable x and its truth value depends on the value that x assumes. We will see later on how this expression can be made into a mathematical statement using the quantifiers, 'For all' and 'There exists'.

Mathematical statements are often indicated using capital letters. For example, we might describe the statement 'five is less than eight' by writing P: five is less than eight. The negation of P, symbolized by $\sim$ P, is the statement having the opposite truth value. That is, when P is true, $\sim$ P is false and when P is false, $\sim$ P is true. For the example P above, $\sim$ P is the statement,''five is not less than eight', or 'five is greater than or equal to eight'.

For an arbitrary mathematical statement P, we can indicate the possible truth values for P and $\sim$ P in the table below, called a truth table.

P	$\sim$ **P**
T	F
F	T

1.2 Compound Statements

In mathematics as in any language, compound statements are formed by combining simpler ones using connectives. The connectives generally used in mathematics are 'and', 'or', 'if ...then', 'if and only if'. The truth value of a compound statement will depend on the truth value of its simpler components.

Definition 1.2: Given two statements P, Q, the compound statement, P *and* Q, called the conjunction, is denoted by $P \wedge Q$ and is defined by the following truth table.

P	**Q**	**P** $\wedge$ **Q**
T	T	T
T	F	F
F	T	F
F	F	F

Note that the conjunction, $P \wedge Q$, is true only when both P and Q are true.

Example 1.1: If P, Q are the statements P: Salt Lake City is in Utah, Q: Las Vegas is in California, then the statement, $P \wedge Q$: Salt Lake City is in Utah and Las Vegas is in California, is false since 'Las Vegas is in California' is a false statement.

The two statements P, Q can also be combined using the connective 'or' as in P *or* Q. This connective has a different meaning in mathematics than when it is used in the english sentence, 'Today I will go to school or I will ski all day'. Here this means that I will do one or the other of these two actions but not both. The word 'or' used in this sense is called the 'exclusive or'. The sentence, 'Today I will read a book or take a nap', allows for the possibility that I could read a book, or take a nap, or read a book and take a nap. The word ‘or’ used in this way is called the ‘inclusive or’ and this is the only use of the connective ‘or’ in mathematics.

Definition 1.3: The statement P *or* Q, called the disjunction and denoted by $P \vee Q$, is defined by the truth table table below.

P	**Q**	**P ∨ Q**
T	T	T
T	F	T
F	T	T
F	F	F

Notice that P *or* Q is true if at least one of the statements is true.

Example 1.2: Consider the two statements, P: 5 is a prime number, Q: 7 is an even number. Since P is true, the disjunction, P or Q: ‘5 is a prime number or 7 is an even number’, is a true statement.

The statement, ‘If the day is Tuesday, then Mary is in school’ uses the connective, if...then, to combine the two statements P; the day is Tuesday, Q: Mary is in school. This type of compound statement is called an implication and is denoted by $P \Rightarrow Q$.The truth table for the implication is not as intuitive as the previous truth tables. However, if we consider this statement as a promise, then the only time the promise is broken, or the implication is false, is if the day is Tuesday and Mary is not in school. That is, the only time the statement is false is if P is true and Q is false. Note that if the day is not Tuesday, Mary may or may not be in school and the promise about what happens on Tuesday is not broken. With this reasoning we make the following definition.

Definition 1.4: The statement, *If* P, *then* Q, called an implication and denoted by $P \Rightarrow Q$, is defined by the truth table below.

P	**Q**	**P ⇒ Q**
T	T	T
T	F	F
F	T	T
F	F	T

Note carefully that the only time this implication is false is when P is true and Q is false.

Example 1.3: Consider the following implications.

i. If $3 = 5$, then $9 = 25$.

ii. If $3 = 5$, then 4 is even.

iii. If the square of a natural number is even, then the number itself is even.

iv. If the sum of two natural numbers is even, then both numbers must be even.

The first two implications, are of the form P $\Rightarrow$ Q where P: $3 = 5$ is a false statement. So, the first two implications are true. Notice that in the first, Q is false whereas in the second, Q is true.

In the third implication, both P and Q are true statements, so the implication, P $\Rightarrow$ Q, is a true statement. The fourth implication is false since 3, and 5 have a sum of 8, an even number, yet neither 3, nor 5 are even. In this example, P is true but Q is false.

The last connective to consider is the biconditional statement, P *if and only if* Q as in the statement, I can get a refund if and only if I have my receipt.

Definition 1.5: The biconditional, P *if and only if* Q, denoted by P $\Leftrightarrow$ Q is defined by the truth table below.

P	**Q**	**P $\Leftrightarrow$ Q**
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the biconditional, P $\Leftrightarrow$ Q, is a true statement only when P and Q have the same truth value.

Example 1.4: The biconditional statement, 'A rectangle is a square if and only if all of its sides of equal length', is a true statement whereas the statement, 'A quadrilateral is a square if and only if the sides of the quadrilateral are of equal length' is a false statement since a rhombus is a quadrilateral with sides of equal length that is not necessarily a square.

Section 2. Equivalent Statements, Negating Statements

2.1 Equivalent Statements

It is reasonable to think that the biconditional, P $\Leftrightarrow$ Q is in some way equivalent to the statement, (P $\Rightarrow$ Q) $\wedge$ (Q $\Rightarrow$ P).

Definition 2.1: Suppose S and T are two compound statements formed from the simple statements P, Q. The statements S and T are said to be equivalent if their truth values are the same for all possible combinations of truth values of P, Q. In that case we write, S ≡ T.

Looking at the truth table below we see that according to this definition,
$P \Leftrightarrow Q \equiv [(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$.

P	**Q**	$\mathbf{P \Rightarrow Q}$	$\mathbf{Q \Rightarrow P}$	$\mathbf{P \Leftrightarrow Q}$	$\mathbf{(P \Rightarrow Q) \wedge (Q \Rightarrow P)}$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Using truth tables, we can, in a straightforward way, determine whether or not two statements of interest are equivalent. For example, although it may not be immediately obvious, from the following truth table we see that the two statements, $P \Rightarrow Q$ and $\sim P \vee Q$ are equivalent. We will see that it is useful to be able to express the implication, $P \Rightarrow Q$ in terms of the disjunction, $\sim P \vee Q$.

P	**Q**	$\mathbf{P \rightarrow Q}$	$\mathbf{\sim P \vee Q}$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

2.2 Negating Statements

Using the definition of equivalent statements and recalling that ∼ P is that statement that has the opposite truth value from P, we can develop the following rules for negating conjunctions, disjunctions, and implications.

Basic Negation Rules:

i. $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$: This is shown in the next truth table.

P	**Q**	$\mathbf{P \wedge Q}$	$\mathbf{\sim (P \wedge Q)}$	$\mathbf{\sim P \vee \sim Q}$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

This means that the negation of 'P and Q' is the statement, 'not P or not Q' where 'or' is the inclusive 'or'.

Example 2.1: For the two statements, P: Salt Lake City is in Utah, Q: Las Vegas is in California, $P \wedge Q$ is the statement 'Salt Lake City is in Utah and Las Vegas is in California'. The negation of this statement is 'Salt Lake City is not in Utah or Las Vegas is not in California'. Note that in this example, $P \wedge Q$ is false since Q is false so its negation is a true statement.

Consider the negation of the statement, '4 is a prime number and 4 is odd'. The negation is: '4 is not a prime number or 4 is not odd. Here the negation is a true statement since 4 is neither prime, nor odd.

ii. $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$: You should construct the truth table to show this is correct. In words then, the negation of 'P or Q' is the statement 'not P and not Q'.

Example 2.2: Suppose that x is a real number. The negation of the statement 'x ≤ -2 or x ≥ 2' is the statement 'x > -2 and x < 2'. In other words, $-2 < x < 2$.

We should take a minute here to observe that these two negation rules can be summarized by saying: to negate a conjunction of two statements, take the disjunction of the negated statements; to negate a disjunction of two statements, take the conjunction of the negated statements.

iii. $\sim (P \Rightarrow Q) \equiv P \wedge \sim Q$: This is shown in the truth table below.

P	**Q**	**P ⇒ Q**	**∼ (P ⇒ Q)**	**P ∧ ∼ Q**
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

Note here once again that the only time the implication, P $\Rightarrow$ Q, is false and hence $\sim$ (P $\Rightarrow$ Q) is true occurs when (P$\wedge \sim$ Q) is true, that is when P is true and Q is false.

Example 2.3: Consider the infinite series, $\sum_{n=1}^{+\infty} a_n$. In Calculus one shows that, given the two statements, P: $\lim_{n\to+\infty} a_n = 0$, and Q: $\sum_1^{+\infty} a_n$ converges, the implication P $\Rightarrow$ Q is false . The Harmonic series, $1 + 1/2 + 1/3 + ...$ which does not converge provides a counterexample, that is an example where P is true and Q is false.

iv. $\sim$ (P $\Leftrightarrow$ Q) $\equiv$ (P$\wedge \sim$ Q) $\vee$ (Q$\wedge \sim$ P): You should verify this by applying the above negation rules or by constructing the appropriate truth table. Be sure to note that the only way the statement $\sim (P \Leftrightarrow Q)$ can be true and hence $P \Leftrightarrow Q$ false is if P and Q have different truth values.

2.3 Implications and Their Connections

In mathematics, implications and their truth value are very important as many theorems or propositions are stated as implications. Given two statements, P, Q, there are four implications that are important to consider. Suppose we call $P \Rightarrow Q$ the given statement. Then the four implications are:

Statement: $P \Rightarrow Q$

Converse: $Q \Rightarrow P$

Contrapositive: $\sim Q \Rightarrow \sim P$

Inverse: $\sim P \Rightarrow \sim Q$

Example 2.5: Suppose P: it is Tuesday, Q: Mary is in school. Then the above implications become:

Statement: If it Tuesday, then Mary is in school.

Converse: If Mary is in school, then it is Tuesday.

Contrapositive: If Mary is not in school, then it is not Tuesday.

Inverse: If it is not Tuesday, then Mary is not in school.

These four implications are clearly related. In fact, a statement and its contrapositive are equivalent, as are the converse and the inverse. The next table verifies that a statement and its contrapositive are equivalent. You should verify that the converse and inverse of a given statement are equivalent by examining the appropriate truth table.

P	**Q**	$\mathbf{P \Rightarrow Q}$	$\mathbf{\sim Q \Rightarrow \sim P}$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Notice that in the words of Example 2.5, the fact that a statement and its contrapositive are equivalent means that whenever 'If it is Tuesday, then Mary is in school' is a true statement, then if we observe that Mary is not in school, then the day cannot be Tuesday.

Before leaving this section we should take careful note that a statement and its converse are not logically equivalent. That is, [P $\Rightarrow$ Q] is not equivalent to [Q $\Rightarrow$ P]. Or, a statement P $\Rightarrow$ Q can be true, yet its converse, Q $\Rightarrow$ P be false.

Example 2.6 A good example of this is the theorem, 'If a function f is differentiable at x = 0, then f is continuous at x = 0.', is a true statement. However, its converse, 'If a function f is continuous at x = 0, then f is differentiable at x = 0', is false which is demonstrated by the function f(x) = |x|.

3. Methods of Proof

Theorems and propositions in mathematics are essentially implications about mathematical concepts stated in the form P $\Rightarrow$ Q. Usually we call P the hypothesis and Q the conclusion. How can one determine if the statement, P $\Rightarrow$ Q, is true, or in other words, the hypothesis implies the conclusion?

Remember from the truth table of this implication, that the only time this implication is **false** is when P is true and Q is false. Hence to prove that this implication is true we must show that P being true forces Q to be true as well. How do we that? We will consider three methods of proof below.
3.1 Direct Proof. In this method we start by assuming P (the hypothesis) is true, then we build a chain of implications, each of which have previously been shown to be true, leading to Q (the conclusion). Suppose there were 6 implications in the chain. Then we could describe the method of direct proof in this way:

$$P, P \Rightarrow P_1, P_1 \Rightarrow P_2, P_2 \Rightarrow P_3, P_3 \Rightarrow Q_1, Q_1 \Rightarrow Q_2, Q_2 \Rightarrow Q.$$

If P and the implications in the above chain are all true, then we must conclude logically that Q is true as well. Note carefully however, if we refer to the truth table for P $\Rightarrow$ Q, knowing that P $\Rightarrow$ Q is a true statement does not say that P itself is true or that Q is true, only that the **implication** is a true statement. Recall that P can be false, Q true or false and still the implication is true.

Before we consider an example, note that when a theorem is written as, 'If P, then Q', P being the hypothesis and Q being the conclusion, to prove the theorem is true, we must show that whenever the hypothesis is true, the conclusion is true as well.

Example 3.1: (example from Mathematical Reasoning by Ted Sundstrom)

Let's examine the implication, 'If x and y are odd integers, then the product xy is an odd integer'. For this proof we need to know that an integer is defined to be odd if it can be expressed as (2n+1) where n is an integer. Consider the following statements:

P: x, y are odd integers (hypothesis)

P_1: There exist integers m,n such that x=2n+1, y=2m+1

P_2: xy=x=(2n+1)(2m+1)=2(2mn+m+n)+1

Q_1: There is an integer k such that xy = 2k+1

Q: xy is an odd integer (Conclusion)

Each of the implications below is true for the reasons given in the the parenthesis after the implication. So, if P is true, this chain of implications demonstrates that Q is true as well.

$P \Rightarrow P_1$ (Definition of odd integer)

$P_1 \Rightarrow P_2$ (Substitution and Algebra)

$P_2 \Rightarrow Q_1$ (k=2mn+m+n is an integer since the integers are closed under multiplication and addition)

$Q_1 \Rightarrow Q$ (Definition of odd integer)

While this is an outline of the structure of the proof, we would write the theorem and proof in words more or less in the manner below.

Theorem: If x and y are odd integers, then the product xy is an odd integer.

Proof: We assume that x and y are odd integers. From the definition of an odd integer, there are integers m,n such that x=2n+1 and y=2m+1. Then the product xy = (2n+1)(2m+1) = 2(mn+n+m) + 1. Because the integers are closed under addition and multiplication, k=(2mn+n+m) is an integer. Since xy = 2k+1, by definition, xy is an odd integer.

3.2 Proof by Contrapositive

This method of proof is simply a direct proof of the contrapositive. That is, if we prove directly that $\sim$ Q $\Rightarrow \sim$ P is a true statement, since this implication is equivalent to P $\Rightarrow$ Q, then we have also shown that P $\Rightarrow$ Q is true. Sometimes the contrapositive is easier to prove than the given statement.

3.3 Proof by Contradiction

A third method of proof relies on finding a contradiction, that is, a statement that is always false. For example, given a statement P, the statement (P$\wedge \sim$ P) is a contradiction.

A proof by contradiction is based on the fact that P $\Rightarrow$ Q is equivalent to ($\sim$ P $\vee$ Q), an equivalency demonstrated earlier. Recall that the negation of ($\sim$ P $\vee$ Q) is (P$\wedge$ $\sim$ Q). If we can show that this negation is false, then the statement, ($\sim$ P $\vee$ Q), is true and hence so is the equivalent statement, P $\Rightarrow$ Q.

Suppose we could show that for some contradiction C, (P$\wedge$ $\sim$ Q) $\Rightarrow$ C is a true implication. (This would mean that whenever (P$\wedge$ $\sim$ Q) holds true, we are led to a contradiction.) Since C is a contradiction, it is always false. Hence for the implication to be true, we must have (P$\wedge$ $\sim$ Q) false as well. But as we said above, if (P$\wedge$ $\sim$ Q) is false, its negation, ($\sim$ P $\vee$ Q) must be true and the equivalent statement, P $\Rightarrow$ Q, is true as well.

Example 3.2: Suppose x is a real number and consider the theorem: If $x > 0$, then $\frac{1}{x} > 0$. Let P, Q be the statements, P: $x > 0$, Q: $\frac{1}{x} > 0$. The theorem we are considering is the implication, P $\Rightarrow$ Q. We will show below that (P$\wedge$ $\sim$ Q) leads to, or implies, a contradiction and hence the theorem is true.

Assume that $[x > 0$ and $\frac{1}{x} \leq 0]$ is a true statement. Since $x > 0$, we can multiply on both sides of the inequality $\frac{1}{x} \leq 0$ by x. Doing this, we obtain the inequality, $x(\frac{1}{x}) \leq x(0)$, or $1 \leq 0$. That is, we have shown by direct proof that, the implication, $[x > 0$ and $\frac{1}{x} \leq 0] \Rightarrow [1 \leq 0]$ is a true statement. Clearly the inequality, $1 \leq 0$, is a contradiction, say C. Since C is false, the only way our implication can be true is if the statement, '$[x > 0$ and $\frac{1}{x} \leq 0]$' is false. Then whenever $x > 0$, we must have $\frac{1}{x} > 0$.

In a subtle way, the above argument used a mathematical sentences called predicate and a quantifier, two topics we consider in the next section.

4. Predicates and Quantifiers

Mathematical sentences such as 'x+3 = -10', or 'x > 0' , or 'x <y' are not mathematical statements since whether they are true or false depends on the value of the variable x. Such sentences are called **predicates** and are denoted by P(x) when only the variable x is involved, or by P(x,y) when the two variables x, y are involved.

One way to make a predicate into a statement is to specify particular value(s) for the variable(s). So, in 'x+3 = -10', if x= 2 we have a false statement whereas if x= -13, we have a true statement. Another way to turn a predicate into a statement is to use quantifiers.

4.1 Two Quantifiers

The sentence, 'For all real x, x+3 = -10' is a mathematical statement that is false, whereas the sentence, 'There exists a real x such that x+3 = -10' is a true statement. Letting P(x) be the predicate, x+3 = -10, we can write these staments as 'For all real x, P(x)', and 'There is a real x such that P(x)'. Generally we assume that there is a specific set of numbers or objects under discussion called the universal set. If our discussion focuses solely on the real numbers, then we can write the above statements more simply as, 'For all x, P(x)' and 'There is an x such that P(x)'.

Definition 4.1: Universal and Existential Quantifiers

The expression 'For all' is the universal quantifier and is denoted by the symbol, ∀. For the statement '$\forall \text{x}, \text{P(x)}$' to be true, P(x) must be a true statement when x is replaced by any value in the universal set.

The expression 'There exists' is the existential quantifier and is denoted by the symbol, ∃. Using the symbol ∋ as shorthand for the words 'such that', we write the statement, 'There exists an x such that P(x)' as: ∃x ∋ P(x). For this statement to be true, there must be at least one x in the universal set, say x_0 for which P(x_0) is a true statement.

Example 4.1: Given the statement P: x+3 = - 10, and using the above notation we can see that $\forall \text{x}, \text{P(x)}$ is a false statement since for x = 2, P(2) is false. However, $\exists x \ni P(x)$ is a true statement since P(-13) is true.

It is worth noting here that a quantified statement may be true for one universal set but not for another. For example, $\forall x, x > 0$ is true for the set of natural numbers but is not true for the set of real numbers. On the other hand, $\exists x \ni x + 3 = 0$ is false for the set of natural numbers but true for the set of integers. So, in any discussion it must be clearly understand what the universe set is.

4.2 Negating Quantifiers

How do we negate statements involving quantifiers? The statement, 'For all x, P(x)', is true only when P(x) is true for any number in the universal set. So for, 'For all x, P(x)' to be false, there must be some x in the universal set, say x_0 such that P(x_0) is false. In other words, $\sim [\forall \text{x}, \text{P(x)}]$ is the statement, $[\exists x \ni \sim P(x)]$.

Example 4.2: Assume our universal set is the set of real numbers. and consider the statement, $\forall x, x^2 \geq 0$. The negation of this statement is, $\exists x \ni x^2 < 0$. Here the original statement was true and its negation is false.

Similarly, if there is no x such that P(x) is true, then for all x, P(x) must be false. So we see that the negation of $\exists x \ni P(x)$ is the statement, $\forall, \sim P(x)$.

Example 4.3: Assume our universal set is the set of real numbers. The negation of, $\exists x \ni |x| < 0$ is the statement, $\forall x, |x| \geq 0$. Since the original statement was false, its negation is true.

We summarize both of these negations involving quantifiers with the following rule.

Basic Rules for Negating Quantifiers:

i.$\sim [\forall \mathrm{x}, \mathrm{P(x)}] \equiv [\exists \mathrm{x} \ni \sim \mathrm{P(x)}]$

ii.$\sim [\exists \mathrm{x} \ni \mathrm{P(x)}] \equiv [\forall \mathrm{x}, \sim \mathrm{P(x)}]$

4.3 Statements with Several Quantifiers and/or Variables

Some statements that we will be interested in will involve more than one quantifier and possibly several variables. We will look at this situation now. When there is more than one quantifier, it is important to note that changing the order in which the quantifiers appear may change the truth value of the statement.

Example 4.4: Let's assume that we are dealing with the real numbers and consider the statement $\forall x\ \exists y \ni\ x + y = 0$. This is a true statement because for y = -x, y is real and x + y = 0. Notice however, that if we change the order of these quantifiers, the resulting statement is $\exists y\ \ni \forall x, x + y = 0$. Notice that this is a quite different statement, one that is in fact false. Changing the order of the quantifiers in this example changes a true statement into one that is false.

To negate complex statements with several quantifiers and several variables we apply in order the rules for negating single quantifiers.

Example 4.5: Consider the statement above, 'For all x, there exists a y, such that x + y = 0'. Letting P(x,y) = x + y = 0, this statement is of the form, $\forall \mathrm{x}\ \exists \mathrm{y}\ \ni \mathrm{P(x, y)}$. Negating this statement in steps we have:

$$\sim [\forall \mathrm{x}\ \exists \mathrm{y} \ni \mathrm{P(x, y)}] = \exists x \ni [\sim \{\exists y \ni P(x, y)\}] = \exists x \ni \forall y, \sim P(x, y).$$

In words, the negation of the original statements is: 'There exists an x such that for all $y, x + y \neq 0$'. Since the original statement was true, the negation is false.

In a similar way, we can derive three more negation rules, that together with the above example, summarize how to negate statements involving two quantifiers and two variables. These four rules are stated below first in symbols.

Negation Rules for 2 quantifiers and 2 variables using symbols:

i. $\sim [\forall \mathrm{x}\ \exists \mathrm{y} \ni \mathrm{P(x, y)}] \equiv \exists x \ni \forall y, \sim P(x, y)]$.

ii. $\sim [\forall \mathrm{x}\ \forall \mathrm{y}, \mathrm{P}(\mathrm{x}, \mathrm{y})] \equiv [\exists x\ \exists y, \sim P(x, y)]$

iii. $\sim [\exists \mathrm{x} \ni \forall \mathrm{y}, \mathrm{P}(\mathrm{x}, \mathrm{y})] \equiv [\forall x\ \exists y, \sim P(x, y)]$

iv. $\sim [\exists \mathrm{x}\ \exists \mathrm{y} \ni \mathrm{P}(\mathrm{x}, \mathrm{y})] \equiv [\forall x\ \forall y, \sim P(x, y)]$

Working with symbols, we can negate statements mechanically using the above rules, even statements involving more than two quantifiers or two variables. However, it is very important to understand the reasoning behind these rules and to be able to express these negations correctly in words. For this reason we will write these four rules below in words and consider an example for each rule.

Negation Rules for 2 quantifiers and 2 variables using words:

i. $\sim$ [For all x there exists a y such that P(x,y)] = [There exists an x such that for all y, not P(x,y)].

Example 4.6: The negation of [For all real x, there exists a natural number n such that $x < n$] is the statement, [There is some real x such that for all natural numbers n, $x \geq n$].

ii. $\sim$ [For all x, for all y, P(x,y)] = [There is some x and there is some y such that, not P(x,y)].

Example 4.7: The negation of [For all real x and for all real y, the product xy is real.] is the statement, [There is some real x and some real y such that the product xy is not real.]

iii. $\sim$ [There is an x such that for all y, P(x,y)] = [For all x, there is some y such that not P(x,y)].

Example 4.8: Taking the negation of [There is some integer n such that for any integer m, m+n = m.] we get the statement, [For all integers n there is some integer m such that $m + n \neq m$.]

iv. $\sim$ [There is some x and there is some y such that P(x,y)] = [For all x and for all y, not P(x,y)]

Example 4.9: The negation of (There exist natural numbers a and b such that $\sqrt{2} = a/b$] is the statement, [For all natural numbers a, b, $a/b \neq \sqrt{2}$.]

We will finish this section with a brief look at the definition of the limit of a sequence of real numbers.

Definition 4.2: The limit of a sequence of real numbers $\{a_n\}$ is the real number a if for every $\epsilon > 0$ there is a real number N such that $|a_n - a| < \epsilon$ whenever n > N.

This definition involves the statement, $\forall\epsilon > 0, \exists \text{N} \ni [(\text{n} > \text{N}) \Rightarrow (|a_n - a| < \epsilon)$. So in order to show that a real number a is the limit of a given sequence $\{a_n\}$, we would need to prove that the above statement is true.

On the other hand, to show a real number a is not the limit of the given sequence $\{a_n\}$ we would need to prove that the above statement is false, or that its negation is true. Applying the above rules in order, we see that the negation is the statement, $\exists\epsilon > 0 \ni \forall \text{N}, \sim [\text{if n} > \text{N}, \text{then} |a_n - a| < \epsilon]$. This statement is equivalent to the statement, $\exists\epsilon > 0 \;\ni \forall \text{N}, [\exists \text{n} > \text{N} \ni |a_n - a| \geq \epsilon]$.

In words this means that we can show a is not the limit of the given sequence $\{a_n\}$ if we can demonstrate that there is some ϵ such that for any N, there is always some n > N for which $|a_n - a| \geq \epsilon$.

This completes a brief introduction to some of the basic logic concepts that we will use in the course.