Pseudo-random numbers: a line of code at a time

mostly

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1 What are random numbers good for?

- ❏ Decision making (e.g., coin flip).
- ❏ Generation of numerical test data.
- ❏ Generation of unique cryptographic keys.
- ❏ Search and optimization via random walks.
- ❏ Selection: `quicksort` (C. A. R. Hoare, *ACM Algorithm 64: Quicksort*, Comm. ACM. **4**(7), 321, July 1961) was the first widely-used divide-and-conquer algorithm to reduce an $\mathcal{O}(N^2)$ problem to (on average) $\mathcal{O}(N \lg(N))$. Cf. Fast Fourier Transform (Gauss 1866 (Latin), Runge 1906, Danielson and Lanczos (crystallography) 1942, Cooley-Tukey 1965). See Figure 1.
- ❏ Simulation.
- ❏ Sampling: unbiased selection of random data in statistical computations (opinion polls, experimental measurements, voting, Monte Carlo integra-

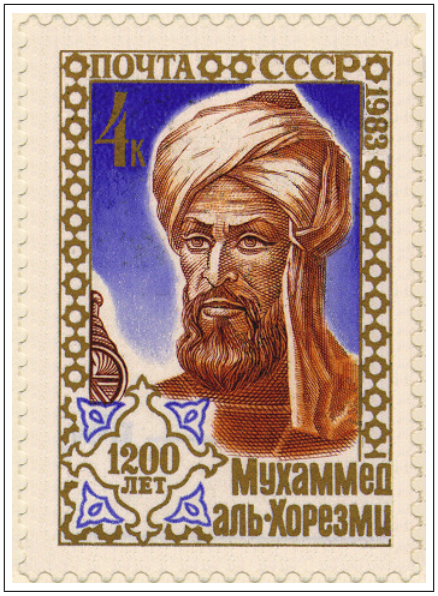
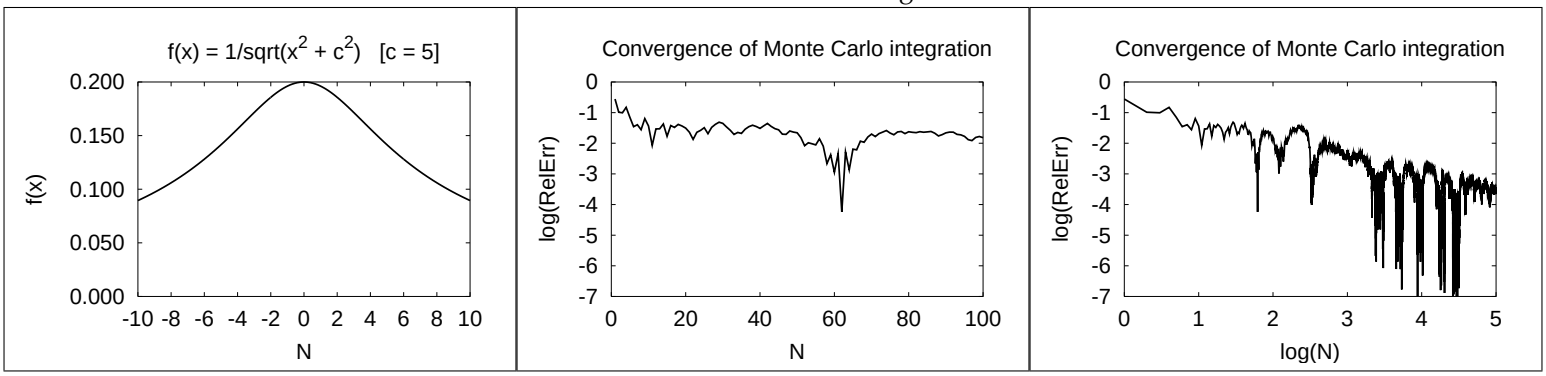


Figure 1: Abu 'Abd Allah Muhammad ibn Musa al-Khwarizmi (ca. 780–850) is the father of *algorithm* and of *algebra*, from his book *Hisab Al-Jabr wal Mugabalah (Book of Calculations, Restoration and Reduction)*. He is celebrated in this 1200-year anniversary Soviet Union stamp.

tion, ...). The latter is done like this:

$$\int_a^b f(x)\,dx \approx \left(\frac{(b-a)}{N}\sum_{k=1}^{N} f(x_k)\right) + \mathcal{O}(1/\sqrt{N}) \quad (x_k \text{ random in } (a,b))$$

Here is an example of a simple, smooth, and exactly integrable function, and the relative error of its Monte Carlo integration.



2 One-time pad encryption

```
% hoc -q crypto.hoc
*******************************************************************************
*******************************************************************************
** Demonstration of a simple one-time pad symmetric-key encryption algorithm **
*******************************************************************************
*******************************************************************************

------------------------------------------------------------------------
The encryption does not reveal message length, although it DOES reveal
common plaintext prefixes:

encrypt(123,"A")

        2b04aa0f ef15ce59 654a0dc6 ba409618 daef6924 5729580b af3af319 f579b0bc


encrypt(123,"AB")

        2b47315b 22fdc9f1 b90d4fdb 1eb8302a 4944eddb e7dd1bff 8d0d1f10 1e46b93c


encrypt(123,"ABC")

        2b47752c 286a4724 40bf188f c08caffa 1007d4cc 2c2495f9 cd999566 abfe0c2d


encrypt(123,"ABCD")

        2b477571 f970b4a2 7346ca58 742e8379 e0ce97b3 1d69dc73 c7d921dc 018bc480


------------------------------------------------------------------------
The encryption does not reveal letter repetititions:

encrypt(123,"AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA")

        2b46736e 3b83cd28 777d88c8 ad1b12dc c28010ef 407d3513 e1ed75bc 5737fd71
        6e68fb7d 4ac31248 94f21f9f d009455f 6d299f


------------------------------------------------------------------------
Now encrypt a famous message from American revolutionary history:

ciphertext = encrypt(123, \
```

```
    "One if by land, two if by sea: Paul Revere's Ride, 16 April 1775")
println ciphertext

        3973974d 63a8ac49 af5cb3e8 da3efdbb f5b63ece 68a21434 19cca7e0 7730dc80
        8e9c265c 5be7476c c51605d1 af1a6d82 9114c057 620da15b 0670bb1d 3c95c30b
        ed

------------------------------------------------------------------------
Attempt to decrypt the ciphertext with a nearby key.  Decryption DOES
reveal the message length, although that flaw could easily be fixed:

decrypt(122, ciphertext)
?^?/?)?D?fN&???w??V???Gj5?????(????1???J???i?i)y?I?-G?????b?o??X?

------------------------------------------------------------------------
Attempt to decrypt the ciphertext with the correct key:

decrypt(123, ciphertext)
One if by land, two if by sea: Paul Revere's Ride, 16 April 1775

------------------------------------------------------------------------
Attempt to decrypt the ciphertext with another nearby key:

decrypt(124, ciphertext)
??$???W?????N????????!?Z?U???????Q??????3?B}`<?O ?P5%??VdNv??kS??

------------------------------------------------------------------------

% cat encrypt.hoc
### -*-hoc-*-
### ======================================================================
### Demonstrate a simple one-time-pad encryption based on a
### pseudo-random number generator.
### [23-Jul-2002]
### ======================================================================

### Usage: encrypt(key,plaintext)
### The returned string is an encrypted text stream: the ciphertext.
func encrypt(key,plaintext) \
{
    plaintext = (plaintext char(255))           # add message terminator
    while (length(plaintext) < 32) \
        plaintext = (plaintext char(randint(1,255))) # pad to 32*n characters
    setrand(key)                                # restart the generator
    n = 0
    ciphertext = "\n\t"
```

```
    for (k = 1; k <= length(plaintext); ++k) \
    {
        ## Output 32-character lines in 4 chunks of 8 characters each

        if ((n > 0) && (n % 32 == 0)) \
            ciphertext = ciphertext "\n\t" \
        else if ((n > 0) && (n % 4 == 0)) \
            ciphertext = ciphertext " "

        ciphertext = sprintf "%s%02x", ciphertext, \
                ((ichar(substr(plaintext,k,1)) + randint(0,255)) % 256)
        n++
    }
    ciphertext = ciphertext "\n"
    return (ciphertext)
}

% cat decrypt.hoc
### -*-hoc-*-
### ==========================================================================
### Demonstrate a simple one-time-pad decryption based on a
### pseudo-random number generator.
### [23-Jul-2002]
### ==========================================================================

### Usage: isprint(c)
### Return 1 if c is printable, and 0 otherwise.
func isprint(c) \
{
    return ((c == 9) || (c == 10) || ((32 <= c) && (c < 127)))
}

__hex_decrypt = "0123456789abcdef"

### Usage: decrypt(key,ciphertext)
### Return the decryption of ciphertext, which will be the original
### plaintext message if the key is correct.
func decrypt(key,ciphertext) \
{
    global __hex_decrypt
    setrand(key)
    plaintext = ""
    for (k = 1; k < length(ciphertext); k++) \
    {
        n = index(__hex_decrypt,substr(ciphertext,k,1))
        if (n > 0) \
```

```
        {                               # have hex digit: decode hex pair
            k++
            c = 16 * (n - 1) + index(__hex_decrypt,substr(ciphertext,k,1)) - 1
            n = int((c + 256 - randint(0,255)) % 256) # recover plaintext char
            if (n == 255) \
                break;
            if (!isprint(n)) \
                n = ichar("?")  # mask unprintable characters
            plaintext = plaintext char(n)
        }
    }
    return (plaintext)
}
```

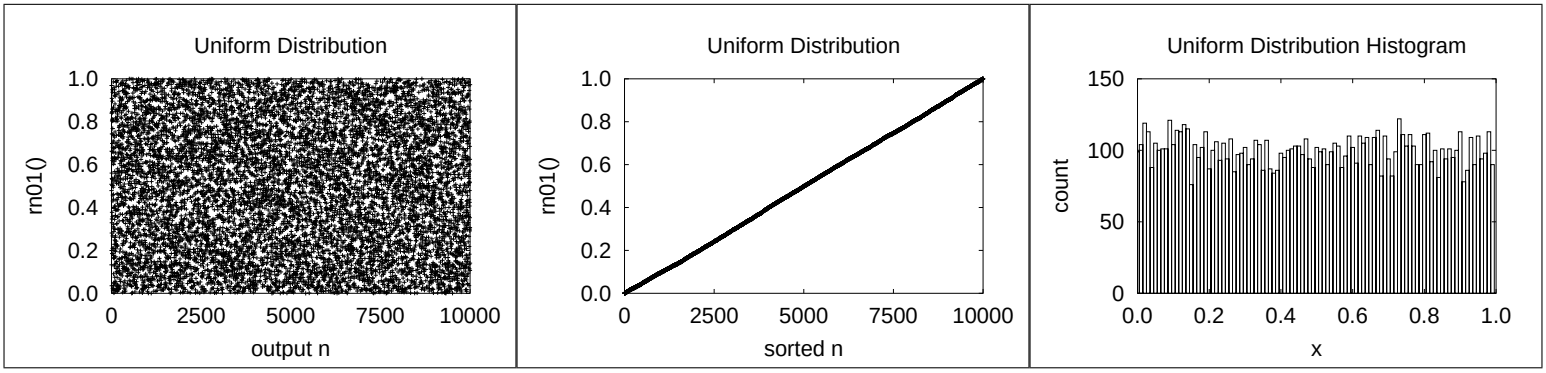
3 When is a sequence of numbers random?

If the numbers are not random, they are at least higgledy-piggledy.
— George Marsaglia (1984)

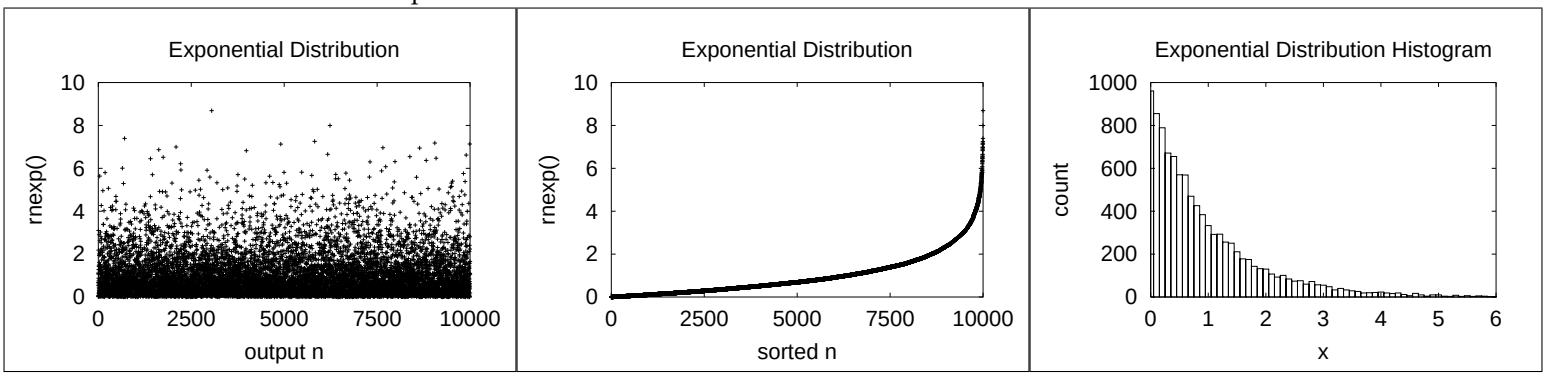
- Observation: all finite computer numbers (both fixed and floating-point) are rational and of limited precision and range: irrational and transcendental numbers are not represented.
- Most pseudo-random number generators produce a long sequence, called the *period*, of distinct integers: truly random integers would have occasional repetitions. Thus, any computer-generated sequence that has no repetitions is strictly *not* random.
- It isn't enough to conform to an expected distribution: the *order* that values appear in must be haphazard. This means that simple tests of moments (called mean, variance, skewness, kurtosis, ... in statistics) are inadequate, because they examine each value in isolation: tests are needed to examine the sequence itself for chaos.
- Mathematical characterization of randomness is possible, but difficult: pp. 149–193 of Donald E. Knuth's *The Art of Computer Programming, vol. 2*.
- The best that we can usually do is *compute statistical measures of closeness* to particular expected distributions. We examine a particularly-useful measure in Section 5.

4 Distributions of pseudo-random numbers

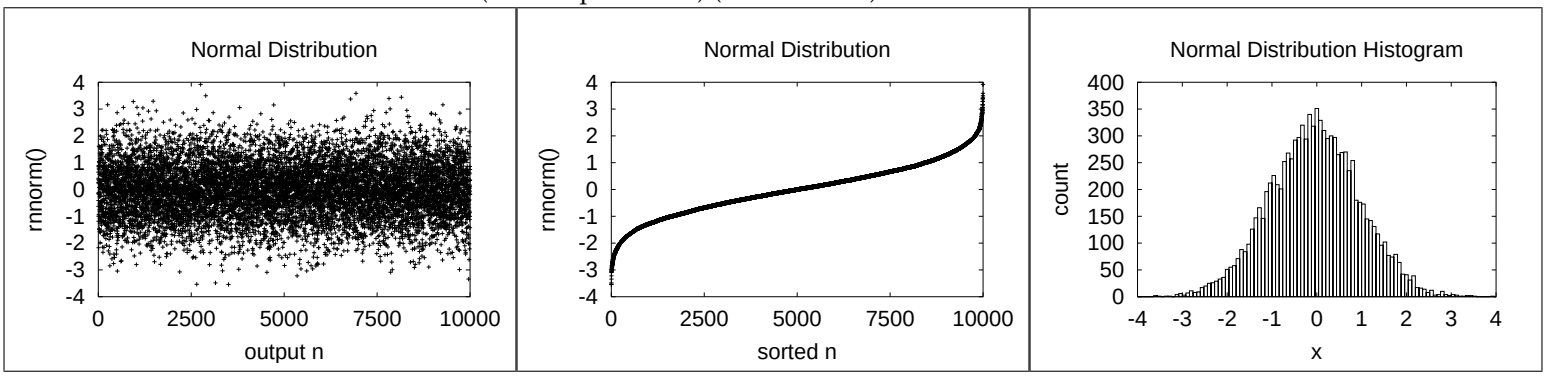
- Uniform (most common).



❏ Exponential.



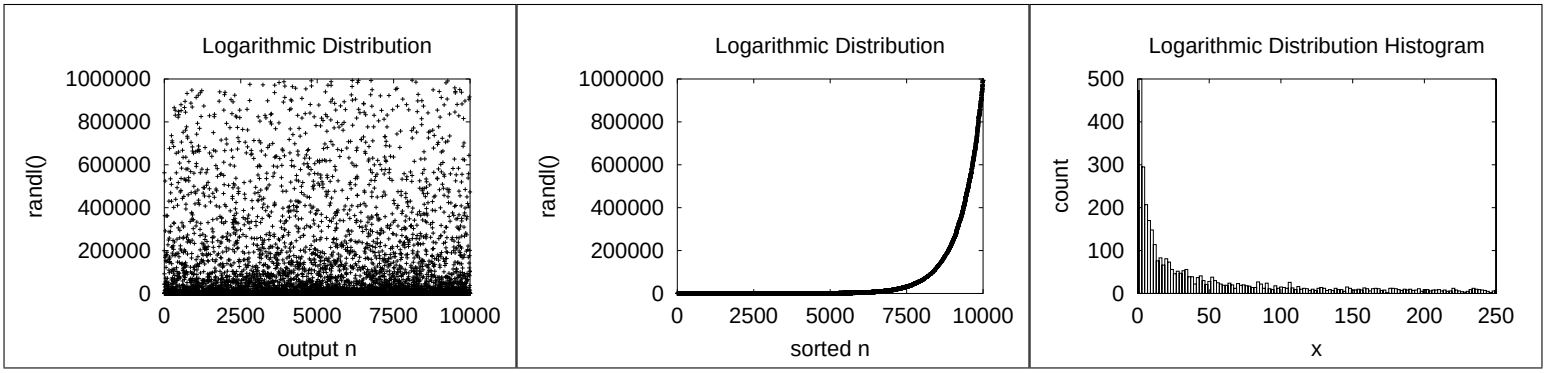
❏ Normal (bell-shaped curve) (see Section 7).



❏ Logarithmic: if ran() is uniformly-distributed in (a, b), define randl$(x) =$ exp$(x$ ran$())$. Then a randl$(\ln(b/a))$ is logarithmically distributed in (a, b).

```
% hoc
a = 1
b = 1000000
```

```
for (k = 1; k <= 10; ++k) \
    printf "%16.8f\n", a*randl(ln(b/a))
    664.28612484
 199327.86997895
 562773.43156449
  91652.89169494
     34.18748767
    472.74816777
     12.34092778
      2.03900107
  44426.83813202
     28.79498121
```



5 Goodness of fit: the χ^2 measure

Given a set of *n independent observations* with measured values M_k and expected values E_k, then $\sum_{k=1}^{n} |(E_k - M_k)|$ is a measure of goodness of fit. So is $\sum_{k=1}^{n} (E_k - M_k)^2$. Statisticians use instead a measure introduced by Pearson (1900):

$$\chi^2 \text{ measure} = \sum_{k=1}^{n} \frac{(E_k - M_k)^2}{E_k}$$

Equivalently, if we have s categories expected to occur with probability p_k, and if we take n samples, counting the number Y_k in category k, then

$$\chi^2 \text{ measure} = \sum_{k=1}^{s} \frac{(np_k - Y_k)^2}{np_k}$$

The theoretical χ^2 distribution depends on the number of degrees of freedom, and table entries look like this (boxed entries are referred to later):

D.o.f.	$p = 1\%$	$p = 5\%$	$p = 25\%$	$p = 50\%$	$p = 75\%$	$p = 95\%$	$p = 99\%$
$\boxed{\nu = 1}$	0.00016	0.00393	0.1015	$\boxed{0.4549}$	1.323	3.841	$\boxed{6.635}$
$\nu = 5$	0.5543	1.1455	2.675	4.351	6.626	11.07	15.09
$\boxed{\nu = 10}$	2.558	3.940	6.737	9.342	12.55	18.31	$\boxed{23.21}$
$\nu = 50$	29.71	34.76	42.94	49.33	56.33	67.50	76.15

This says that, e.g., **for $\nu = 10$, the probability that the χ^2 measure is no larger than 23.21 is 99%**.

For example, coin toss has $\nu = 1$: if it is not heads, then it must be tails.

```
for (k = 1; k <= 10; ++k) print randint(0,1), ""
0 1 1 1 0 0 0 0 1 0
```

This gave four 1s and six 0s:

$$\chi^2 \text{ measure} = \frac{(10 \times 0.5 - 4)^2 + (10 \times 0.5 - 6)^2}{10 \times 0.5} = 2/5 = 0.40$$

From the table, we expect a χ^2 measure no larger than 0.4549 half of the time, so our result is reasonable.

On the other hand, if we got nine 1s and one 0, then we have

$$\chi^2 \text{ measure} = \frac{(10 \times 0.5 - 9)^2 + (10 \times 0.5 - 1)^2}{10 \times 0.5} = 32/5 = 6.4$$

This is close to the tabulated value 6.635 at $p = 99\%$. That is, we should only expect nine-of-a-kind about once in every 100 experiments.

If we had all 1s or all 0s, the χ^2 measure is 10 (probability $p = 0.998$).

If we had equal numbers of 1s and 0s, then the χ^2 measure is 0, indicating an exact fit.

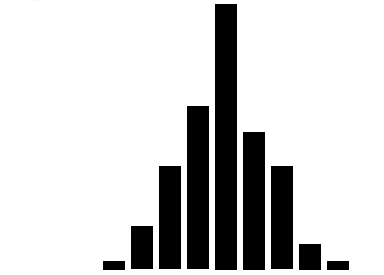
Let's try 100 similar experiments, counting the number of 1s in each experiment:

```
for (n = 1; n <= 100; ++n) \
{sum = 0; for (k = 1; k <= 10; ++k) sum += randint(0,1);
print sum, ""}
4 4 7 3 5 5 5 2 5 6 6 6 3 6 6 7 4 5 4 5 5 4 3 6 6 9 5 3
4 5 4 4 4 5 4 5 5 4 6 3 5 5 3 4 4 7 2 6 5 3 6 5 6 7 6 2
5 3 5 5 5 7 8 7 3 7 8 4 2 7 7 3 3 5 4 7 3 6 2 4 5 1 4 5
5 5 6 6 5 6 5 5 4 8 7 7 5 5 4 5
```

The measured frequencies of the sums are:

100 experiments

k	0	1	2	3	4	5	6	7	8	9	10
				1	1	3	1	1			
Y_k	0	1	5	2	9	1	6	2	3	1	0

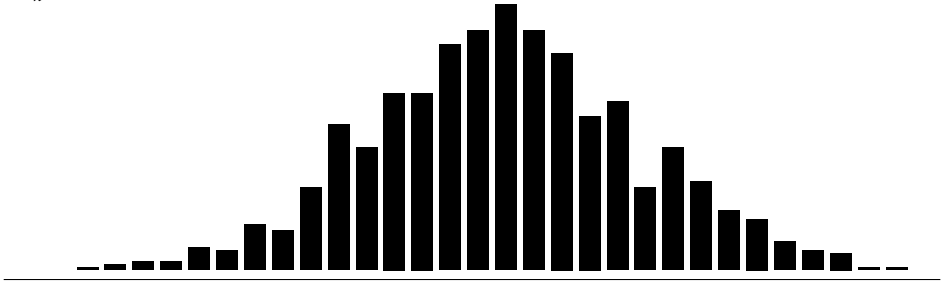


Notice that nine-of-a-kind occurred once each for 0s and 1s, as predicted.

A simple one-character change on the outer loop limit produces the next experiment:

1000 experiments

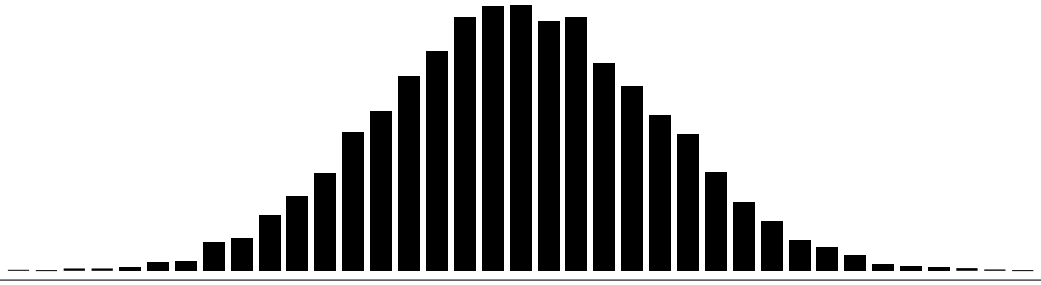
k	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65
							1	1	2	5	4	6	6	7	8	9	8	7	5	5	2	4	3	2	1	1					
Y_k	1	2	3	3	8	7	6	4	9	1	3	2	2	9	4	3	4	6	4	9	9	3	1	1	8	0	7	6	1	1	0



Another one-character change gives us this:

10 000 experiments

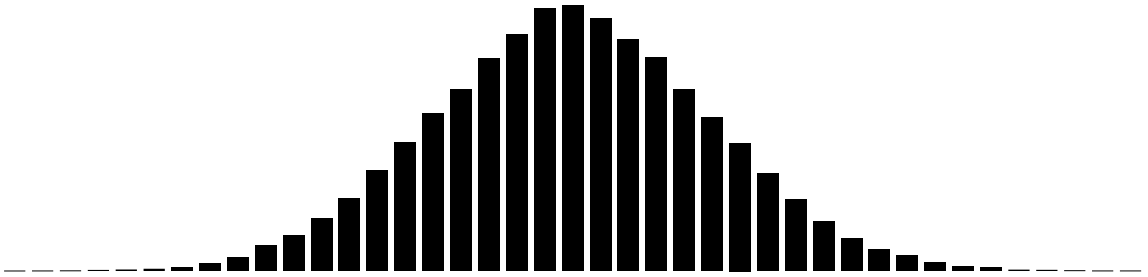
k	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
												1	2	2	4	4	5	6	7	7	8	7	7	6	5	4	4	2	2	1												
							1	2	3	8	9	6	2	9	2	8	8	6	6	9	0	5	6	2	5	7	1	9	0	5	9	7	4	2	1	1					
Y_k	0	0	3	1	7	7	2	7	0	5	9	8	4	5	0	4	8	3	6	9	4	5	6	8	9	0	3	8	7	0	3	0	8	0	5	2	8	4	1	0	0



A final one-character change gives us this result for one million coin tosses:

100 000 experiments

k	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70
											1	1	2	3	3	4	5	6	7	8	8	7	7	6	5	4	3	3	2	1												
							1	2	4	8	0	6	2	1	9	8	6	5	3	1	2	8	1	6	6	7	9	0	2	5	9	6	4	2	1	1					
			1	3	4	7	1	4	1	0	8	3	5	1	8	8	0	8	2	1	2	2	7	0	0	4	6	2	1	4	9	5	7	5	4	0	4	3	2		
Y_k	1	4	2	4	7	0	8	2	8	2	6	3	9	2	7	0	9	7	0	3	7	8	1	7	4	0	2	9	2	4	6	4	4	7	1	7	3	7	1	5	5



In the magazine Science 84 (November 1984), the chi-square test was ranked among the top twenty scientific discoveries of the 20th Century that changed our lives:

1900–1919	1	Bakelite		11	Fission
	2	IQ test		12	Red Shift
	3	Non-Newtonian physics		13	DDT
	4	Blood groups		14	Television
	5	**Chi-square test**	1940–1959	15	Birth control pills
	6	Vacuum tubes		16	Colossus and Eniac
	7	Hybrid corn		17	Pschoactive drugs
	8	Airfoil theory		18	Transistor
1920–1939	9	Antibiotics		19	Double helix
	10	Taung skull		20	Masers and lasers

6 Randomness of digits of π

Here are χ^2 results for the digits of π from recent computational records ($\chi^2(\nu = 9, P = 0.99) \approx 21.67$):

π			
Digits	Base	χ^2	P(χ^2)
6B	10	9.00	0.56
50B	10	5.60	0.22
200B	10	8.09	0.47
1T	10	14.97	0.91
1T	16	7.94	0.46

$1/\pi$

Digits	Base	χ^2	P(χ^2)
6B	10	5.44	0.21
50B	10	7.04	0.37
200B	10	4.18	0.10

Whether the fractional digits of π, and most other transcendentals, are *normal* ($\approx$ equally likely to occur) is an outstanding unsolved problem in mathematics.

De Morgan suspected that Shanks' 1872–73 computation of π to 707 decimal digits was wrong because the frequency of the digit 7 was low. De Morgan was right, but it took a computer calculation by Ferguson in 1946 to show the error at Shanks' digit 528.

7 The Central-Limit Theorem

The
normal
law of error
stands out in the
experience of mankind
as one of the broadest
generalizations of natural
philosophy ◇ It serves as the
guiding instrument in researches
in the physical and social sciences and
in medicine agriculture and engineering ◇
It is an indispensable tool for the analysis and the
interpretation of the basic data obtained by observation and experiment.

— *W. J. Youdon (1956)*
[from Stephen M. Stigler, Statistics on the Table *(1999), p. 415]*

The famous **Central-Limit Theorem** (de Moivre 1718, Laplace 1810, and Cauchy 1853), says:

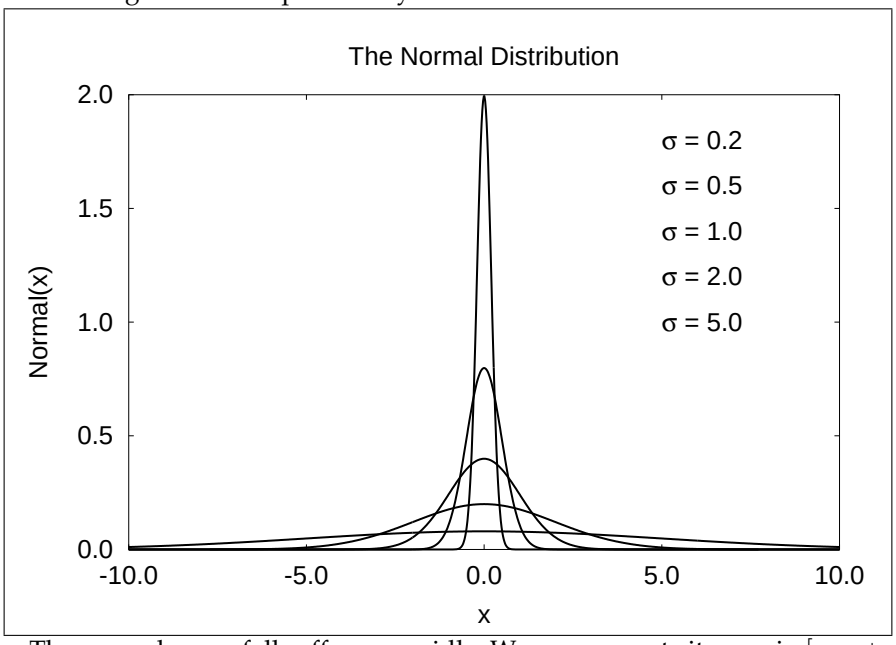
> A suitably normalized sum of independent random variables is likely to be normally distributed, as the number of variables grows beyond all bounds. It is not necessary that the variables all have the same distribution function or even that they be wholly independent.
>
> — I. S. Sokolnikoff and R. M. Redheffer
> *Mathematics of Physics and Modern Engineering*, 2nd ed.

In mathematical terms, this is

$$P(n\mu + r_1\sqrt{n} \le X_1 + X_2 + \cdots + X_n \le n\mu + r_2\sqrt{n}) \approx \frac{1}{\sigma\sqrt{2\pi}} \int_{r_1}^{r_2} \exp(-t^2/(2\sigma^2))dt$$

where the X_k are independent, identically distributed, and bounded random variables, μ is their mean value, and σ^2 is their variance (not further defined here).

The integrand of this probability function looks like this:



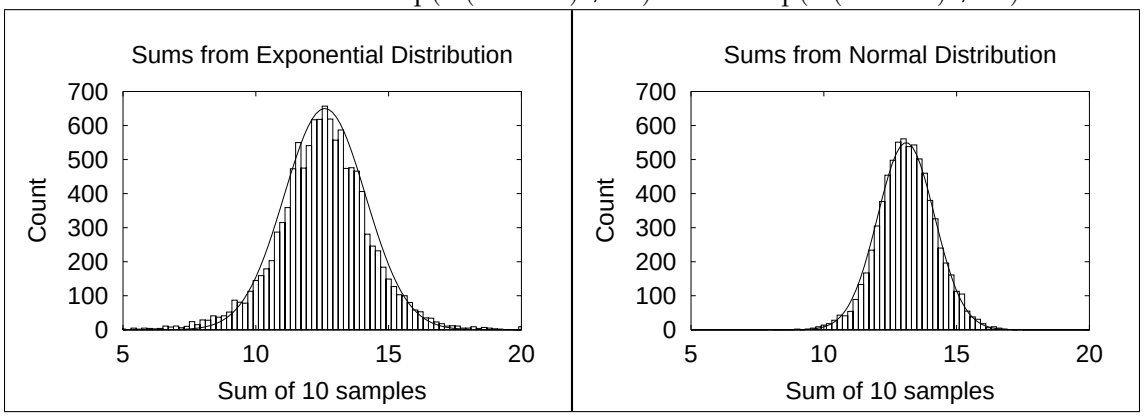
The normal curve falls off very rapidly. We can compute its area in $[-x, +x]$ with a simple midpoint quadrature rule like this:

```
func f(x) {global sigma;
           return (1/(sigma*sqrt(2*PI)))*exp(-x*x/(2*sigma**2))}

func q(a,b){n = 10240; h = (b - a)/n; s = 0;
            for (k = 0; k < n; ++k) s += h*f(a + (k + 0.5)*h);
            return s}

sigma = 3
for (k = 1; k < 8; ++k) printf "%d  %.9f\n", k, q(-k*sigma,k*sigma)

1  0.682689493
2  0.954499737
3  0.997300204
4  0.999936658
5  0.999999427
```

```
6  0.999999998
7  1.000000000
```

In computers, 99.999% (five 9's) availability is 5 minutes downtime per year. In manufacturing, Motorola's 6σ reliability with 1.5σ drift is about 3.4 defects per million (from $q(4.5 * \sigma)/2$).

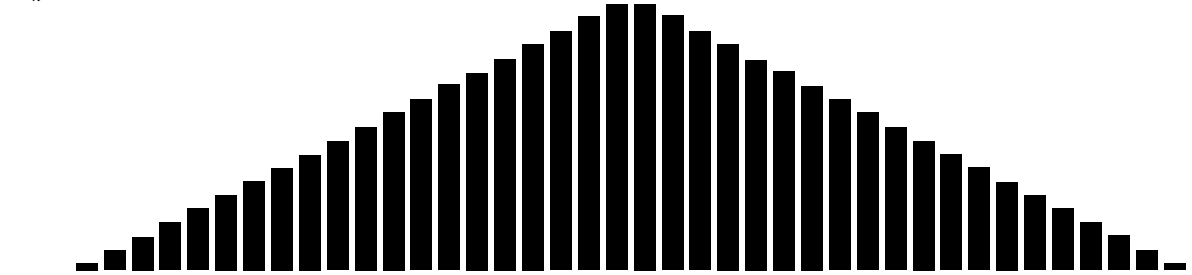
It is remarkable that the Central-Limit Theorem applies also to nonuniform distributions: here is a demonstration with sums from exponential and normal distributions. Superimposed on the histograms are rough fits by eye of normal distribution curves $650 \exp(-(x-12.6)^2/4.7)$ and $550 \exp(-(x-13.1)^2/2.3)$.



Not everything looks like a normal distribution. Here is a similar experiment, using *differences* of successive pseudo-random numbers, bucketizing them into 40 bins from the range $[-1.0, +1.0]$:

10 000 experiments (counts scaled by 1/100)

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
					1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	4	4	4	4	3	3	3	3	2	2	2	2	1	1	1	1				
	1	3	6	8	1	3	6	8	1	3	6	9	1	3	6	8	1	3	6	8	8	6	3	1	8	6	3	1	8	6	3	1	8	6	3	1	8	6	3	1
Y_k	3	5	1	8	3	8	3	7	1	6	2	0	2	9	1	7	4	7	4	7	7	7	7	4	5	5	7	2	8	1	6	2	8	2	7	3	7	3	6	2



This one is known from theory: it is a *triangular* distribution. A similar result is obtained if one takes pair sums instead of differences.

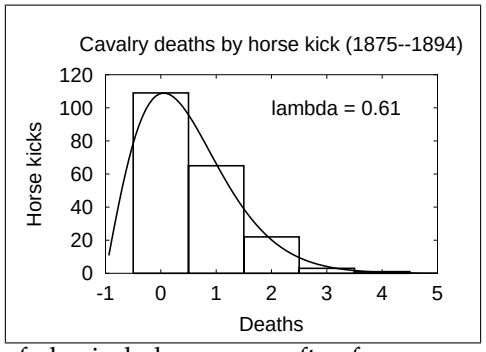
Here is another type, the *Poisson* distribution, which arises in time series when the probability of an event occurring in an arbitrary interval is propor-

tional to the length of the interval, and independent of other events:

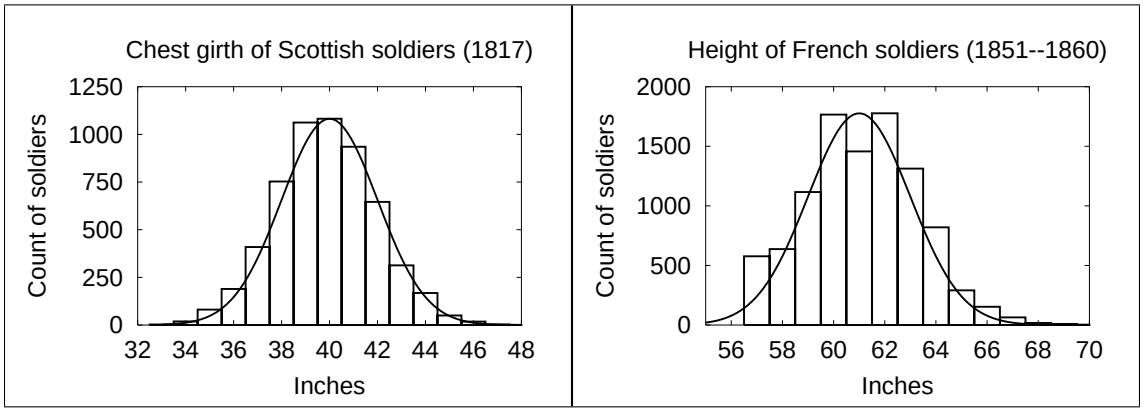
$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

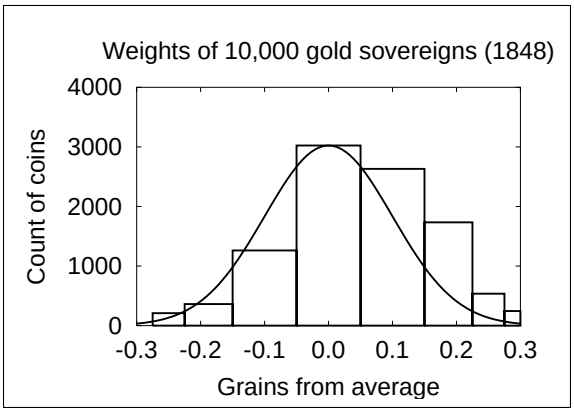
In 1898, Ladislaus von Bortkiewicz collected Prussian army data on the number of soldiers killed by horse kicks in 10 cavalry units over 20 years: 122 deaths, or an average of $122/200 = 0.61$ deaths per unit per year.

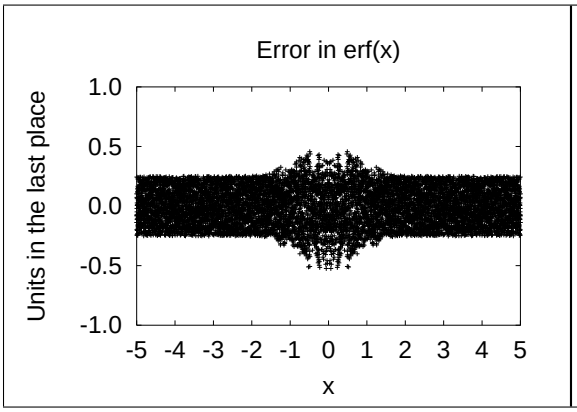
$\lambda = 0.61$		
Deaths	Kicks (actual)	Kicks (Poisson)
0	109	108.7
1	65	66.3
2	22	20.2
3	3	4.1
4	1	0.6

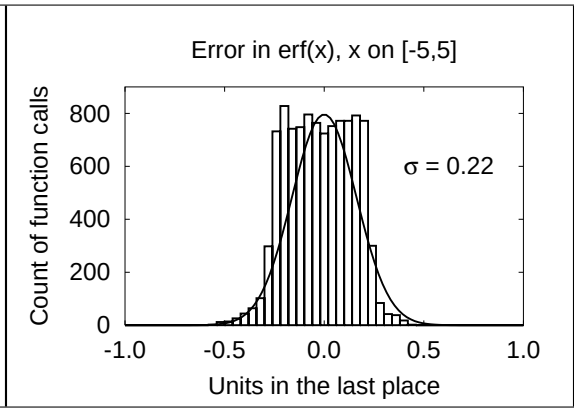


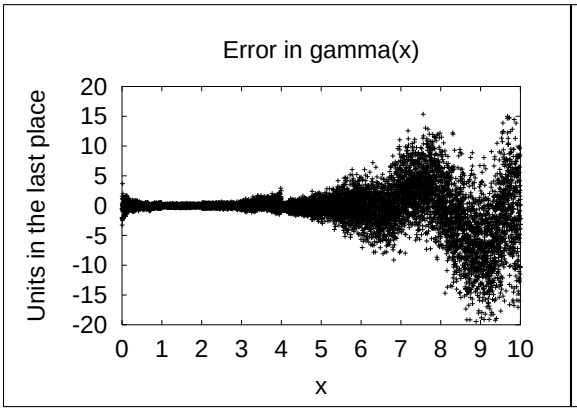
Measurements of physical phenomena often form normal distributions:

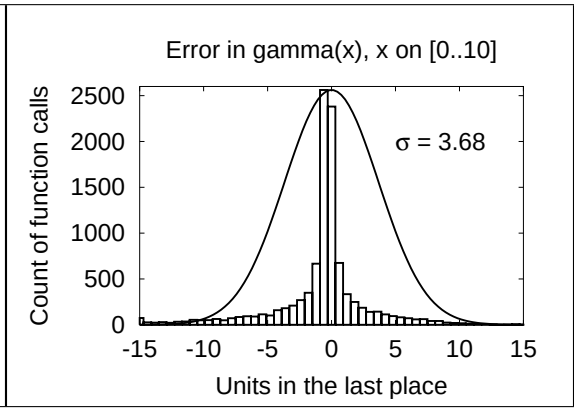


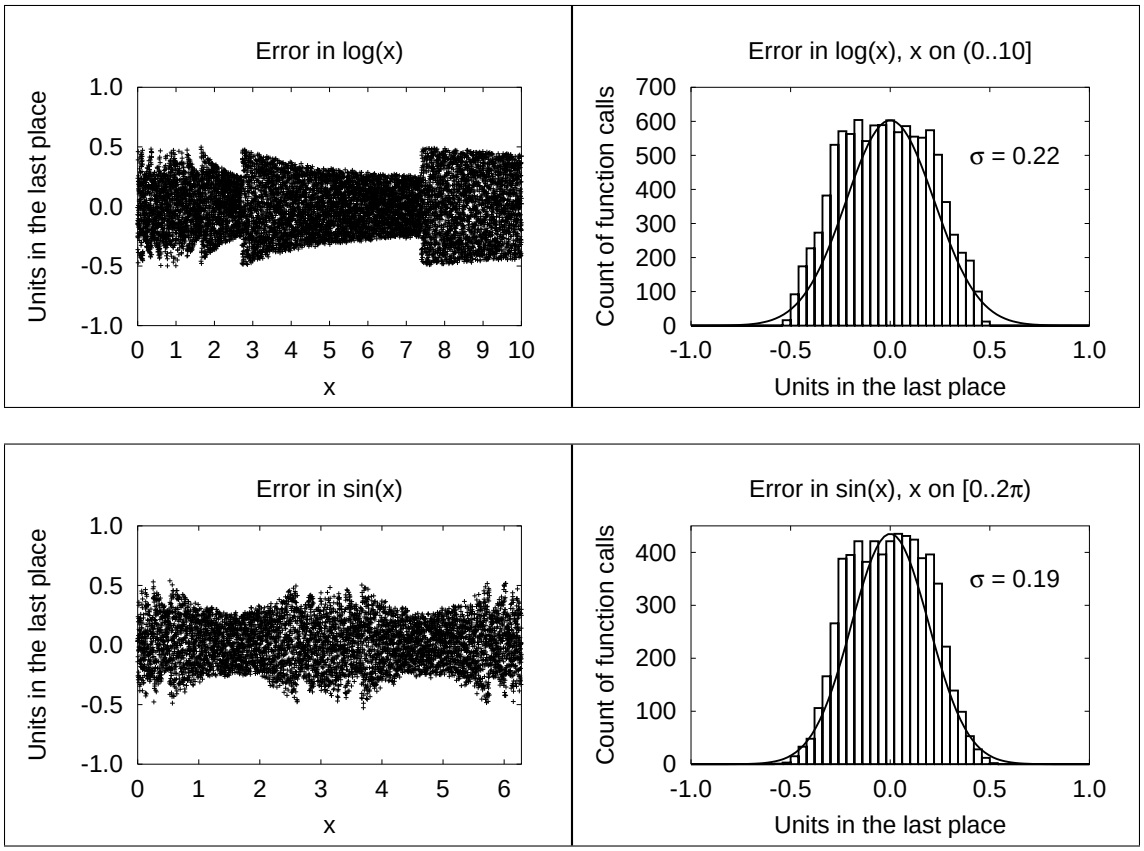

Weights of 10,000 gold sovereigns (1848)
Count of coins
4000
3000
2000
1000
0
-0.3 -0.2 -0.1 0.0 0.1 0.2 0.3
Grains from average


Error in erf(x)
Units in the last place
1.0
0.5
0.0
-0.5
-1.0
-5 -4 -3 -2 -1 0 1 2 3 4 5
x


Error in erf(x), x on [-5,5]
Count of function calls
800
600
400
200
0
σ = 0.22
-1.0 -0.5 0.0 0.5 1.0
Units in the last place


Error in gamma(x)
Units in the last place
20
15
10
5
0
-5
-10
-15
-20
0 1 2 3 4 5 6 7 8 9 10
x


Error in gamma(x), x on [0..10]
Count of function calls
2500
2000
1500
1000
500
0
σ = 3.68
-15 -10 -5 0 5 10 15
Units in the last place



8 How do we generate pseudo-random numbers?

Any one who considers arithmetical methods of producing random numbers is, of course, in a state of sin.
— John von Neumann (1951)
[*The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., p. 1]

He talks at random; sure, the man is mad.
— Margaret, daughter to Reignier, afterwards married to King Henry
in William Shakespeare's *1 King Henry VI*, Act V, Scene 3 (1591)

A random number generator chosen at random isn't very random.
— Donald E. Knuth
[*The Art of Computer Programming, Vol. 2,*

Seminumerical Algorithms, 3rd ed., p. 384]

- ❏ Linear-congruential generators (most common): $r_{n+1} = (ar_n + c) \bmod m$, for integers a, c, and m, where $0 < m$, $0 \le a < m$, $0 \le c < m$, with starting value $0 \le r_0 < m$. Under certain known conditions, the period can be as large as m, unless $c = 0$, when it is limited to $m/4$.
- ❏ Fibonacci sequence (bad!): $r_{n+1} = (r_n + r_{n-1}) \bmod m$.
- ❏ Additive (better): $r_{n+1} = (r_{n-\alpha} + r_{n-\beta}) \bmod m$.
- ❏ Multiplicative (bad): $r_{n+1} = (r_{n-\alpha} \times r_{n-\beta}) \bmod m$.
- ❏ Shift register: $r_{n+k} = \sum_{i=0}^{k-1} (a_i r_{n+i} \pmod 2) \qquad (a_i = 0, 1)$.

Given an integer $r \in [A, B)$, $x = (r - A)/(B - A + 1)$ is on $[0, 1)$.

However, interval reduction by $A + (r - A) \bmod s$ to get a distribution in (A, C), where $s = (C - A + 1)$, is possible only for certain values of s. Consider reduction of $[0, 4095]$ to $[0, m]$, with $m \in [1, 9]$: we get equal distribution of remainders only for $m = 2^q - 1$:

	m	counts of remainders $k \bmod (m+1)$, $k \in [0, m]$									
OK	1	2048	2048								
	2	**1366**	1365	1365							
OK	3	1024	1024	1024	1024						
	4	**820**	819	819	819	819					
	5	**683**	**683**	**683**	**683**	682	682				
	6	**586**	585	585	585	585	585	585			
OK	7	512	512	512	512	512	512	512	512		
	8	**456**	455	455	455	455	455	455	455	455	
	9	**410**	**410**	**410**	**410**	**410**	**410**	409	409	409	409

Samples from other distributions can usually be obtained by some suitable transformation. Here is the simplest generator for the normal distribution, assuming that `randu()` returns uniformly-distributed values on $(0, 1]$:

```
func randpmnd() \
{
    ## Polar method for random deviates
    ## Algorithm P, p. 122, from Donald E. Knuth, The Art
    ## of Computer Programming, 3rd edition, 1998

    while (1) \
    {
        v1 = 2*randu() - 1
        v2 = 2*randu() - 1
        s = v1*v1 + v2*v2
        if (s < 1) break
```

```
    }
    return (v1 * sqrt(-2*ln(s)/s))

    ## (v2 * sqrt(-2*ln(s)/s)) is also normally distributed,
    ## but is wasted, since we only need one return value
}
```

9 Period of a sequence

All pseudo-random number generators eventually reproduce the starting sequence; the *period* is the number of values generated before this happens. Good generators are now known with periods $> 10^{449}$ (e.g., Matlab's `rand()`).

10 Reproducible sequences

In computational applications with pseudo-random numbers, it is *essential* to be able to reproduce a previous calculation. Thus, generators are required that can be set to a given *initial seed*:

```
% hoc
for (k = 0; k < 3; ++k) \
{
    setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000), ""
    println ""
}
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319
88185 5927 13313 23165 64063 90785 24066 37277 55587 62319

for (k = 0; k < 3; ++k) \
{
    ## setrand(12345)
    for (n = 0; n < 10; ++n) print int(rand()*100000), ""
    println ""
}
36751 37971 98416 59977 49189 85225 43973 93578 61366 54404
70725 83952 53720 77094 2835 5058 39102 73613 5408 190
83957 30833 75531 85236 26699 79005 65317 90466 43540 14295
```

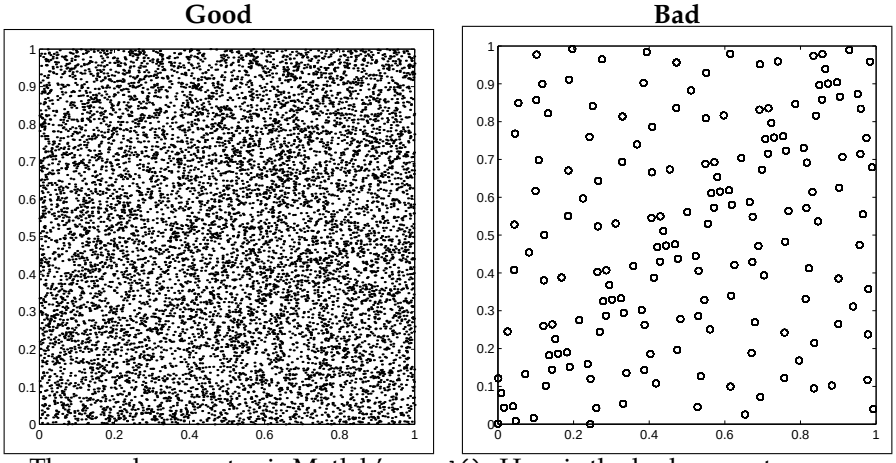
In practice, this means that **software must have its own source-code implementation of the generators**: vendor-provided ones do *not* suffice.

11 The correlation problem

Random numbers fall mainly in the planes
— George Marsaglia (1968)

Linear-congruential generators are known to have correlation of successive numbers: if these are used as coordinates in a graph, one gets patterns, instead of uniform grey. The number of points plotted in each is the same in both graphs:

Good **Bad**



The good generator is Matlab's `rand()`. Here is the bad generator:

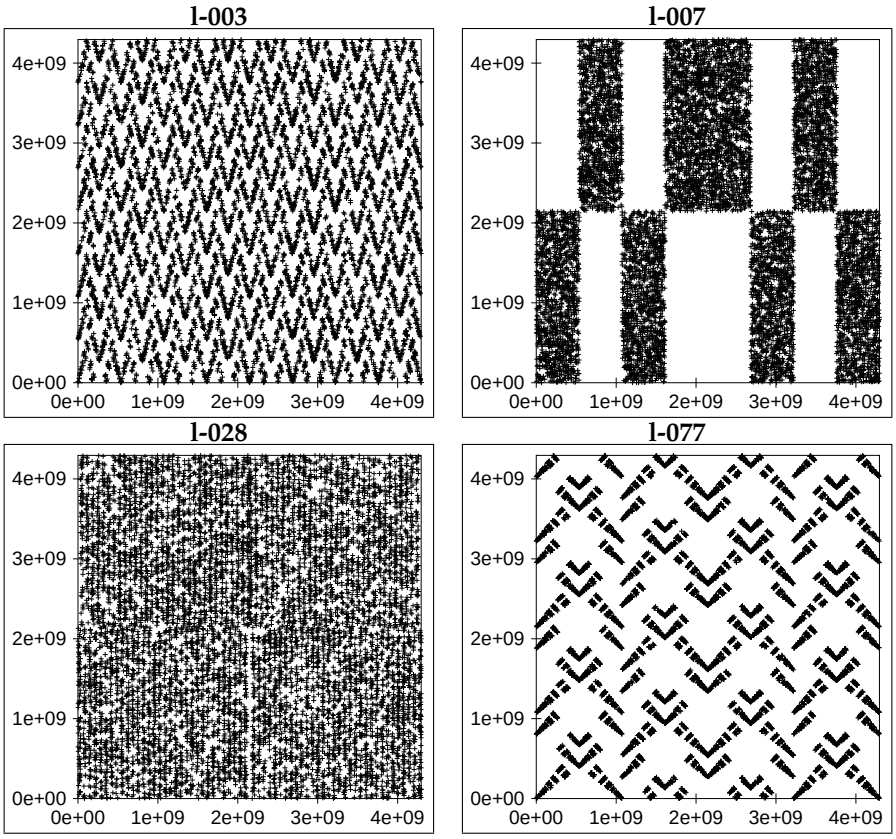
```
% hoc
func badran() { global A, C, M, r; r = int(A*r + C) % M;
                return r }

M = 2^15 - 1
A = 2^7 - 1
C = 2^5 - 1
r = 0
r0 = r
s = -1
period = 0

while (s != r0) {period++; s = badran(); print s, "" }
    31 3968 12462 9889 10788 26660 ...
    22258 8835 7998 0

# Show the sequence period
println period
    175

# Show that the sequence repeats
```

```
for (k = 1; k <= 5; ++k) print badran(), ""
    31 3968 12462 9889 10788
```

12 The correlation problem [cont.]

Marsaglia's (*Xorshift RNGs*, J. Stat. Software **8**(14) 1–6, 2003) family of generators:

```
y ^= y << a; y ^= y >> b; y ^= y << c;
```



13 Generating random integers

When the endpoints of a floating-point uniform pseudo-random number generator are uncertain, generate random integers in [`low`,`high`] like this:

```
func irand(low, high) \
{
    # Ensure integer endpoints
    low = int(low)
```

```
    high = int(high)

    # Sanity check on argument order
    if (low >= high) return (low)

    # Find a value in the required range
    n = low - 1
    while ((n < low) || (high < n)) \
        n = low + int(rand() * (high + 1 - low))

    return (n)
}

for (k = 1; k <= 20; ++k) print irand(-9,9), ""
-9 -2 -2 -7 7 9 -3 0 4 8 -3 -9 4 7 -7 8 -3 -4 8 -4

for (k = 1; k <= 20; ++k) print irand(0, 10^6), ""
986598 580968 627992 379949 700143 734615 361237
322631 116247 369376 509615 734421 321400 876989
940425 139472 255449 394759 113286 95688
```

14 Generating random integers in order

See Chapter 12 of Jon Bentley, *Programming Pearls*, 2nd ed., Addison-Wesley (2000), ISBN 0-201-65788-0. [Published in ACM Trans. Math. Software **6**(3), 359–364, September 1980].

```
% hoc
func bigrand() { return int(2^31 * rand()) }

# select(m,n): select m pseudo-random integers
# from (0,n) in order
proc select(m,n) \
{
    mleft = m
    remaining = n
    for (i = 0; i < n; ++i) \
    {
        if (int(bigrand() % remaining) < mleft) \
        {
            print i, ""
            mleft--
        }
        remaining--
    }
```

```
    println ""
}

select(3,10)
5 6 7

select(3,10)
0 7 8

select(3,10)
2 5 6

select(3,10)
1 5 7

select(10,100000)
7355 20672 23457 29273 33145 37562 72316 84442 88329 97929

select(10,100000)
401 8336 41917 43487 44793 56923 61443 90474 92112 92799

select(10,100000)
5604 8492 24707 31563 33047 41864 42299 65081 90102 97670
```

15 Testing pseudo-random number generators

Most of the tests of pseudo-random number distributions are based on computing a χ^2 measure of computed and theoretical values. **If one gets values $p < 1\%$ or $p > 99\%$ for several tests, the generator is suspect.**

Knuth devotes about 100 pages to the problem of testing pseudo-random number generators. Unfortunately, most of the easily-implemented tests do not distinguish good generators from bad ones. The better tests are much harder to implement.

The Marsaglia Diehard Battery test suite (1985) has 15 tests that can be applied to files containing binary streams of pseudo-random numbers. The Marsaglia/Tsang `tuftest` suite (2002) has only three, and requires only functions, not files, but a pass is believed (empirically) to imply a pass of the Diehard suite. All of these tests produce p values that can be checked for reasonableness.

These tests all expect *uniformly-distributed* pseudo-random numbers. How do you test a generator that produces pseudo-random numbers in some other distribution? You have to figure out a way to use those values to produce an expected uniform distribution that can be fed into the standard test programs. For example, take the negative log of exponentially-distributed values, since $-\log(\exp(-\text{random})) = \text{random}$. For normal distributions, consider succes-

sive pairs (x, y) as a 2-dimensional vector, and express in polar form (r, θ): θ is then uniformly distributed in $[0, 2\pi)$, and $\theta/(2\pi)$ is in $[0, 1)$.

16 Digression: The Birthday Paradox

The *birthday paradox* arises from the question *"How many people do you need in a room before the probability is at least half that two of them share a birthday?"*

The answer surprises most people: it is just 23, not $365/2 = 182.5$.

The probability that *none* of n people are born on the same day is

$$\begin{aligned} P(1) &= 1 \\ P(n) &= P(n-1) \times (365 - (n-1))/365 \end{aligned}$$

The n-th person has a choice of $365 - (n-1)$ days to not share a birthday with any of the previous ones. Thus, $(365 - (n-1))/365$ is the probability that the n-th person is not born on the same day as any of the previous ones, assuming that they are born on different days.

Here are the probabilities that n people *share* a birthday (i.e., $1 - P(n)$):

```
% hoc128
PREC = 3
p = 1; for (n = 1;n <= 365;++n) \
    {p *= (365-(n-1))/365; println n,1-p}
1 0
2 0.00274
3 0.00820
4 0.0164
...
22 0.476
23 0.507
24 0.538
...
30 0.706
...
40 0.891
...
50 0.970
...
70 0.999
...
80 0.9999
...
90 0.999994
...
```

```
100 0.999999693
...
110 0.999999989
...
120 0.99999999976
...
130 0.9999999999962
...
140 0.999999999999962
...
150 0.99999999999999978
...
160 0.9999999999999999900
...
170 0.99999999999999999975
...
180 0.999999999999999999999963
...
190 0.99999999999999999999999967
...
200 0.9999999999999999999999999984
...
210 0.999999999999999999999999999999952
...
365 1.0 - 1.45e-157
366 1.0
```

[Last two results taken from 300-digit computation in Maple.]

17 The Marsaglia/Tsang `tuftest` tests

The first `tuftest` test is the **b'day test**, a generalization of the Birthday Paradox to a much longer year. Here are two reports for it:

Good generator

```
Birthday spacings test: 4096 birthdays, 2^32 days in year
          Table of Expected vs. Observed counts:
Duplicates   0     1     2     3     4     5     6     7     8     9   >=10

Expected   91.6 366.3 732.6 976.8 976.8 781.5 521.0 297.7 148.9  66.2  40.7
Observed    87   385   748   962   975   813   472   308   159    61    30

(O-E)^2/E   0.2   1.0   0.3   0.2   0.0   1.3   4.6   0.4   0.7   0.4   2.8
        Birthday Spacings: Sum(O-E)^2/E= 11.856, p= 0.705
```

Bad generator

```
Birthday spacings test: 4096 birthdays, 2^32 days in year
          Table of Expected vs. Observed counts:
Duplicates   0     1     2     3     4     5     6     7     8     9   >=10

Expected   91.6 366.3 732.6 976.8 976.8 781.5 521.0 297.7 148.9  66.2  40.7
Observed     0     0     0     0     1     3    18    53    82   144  4699

(O-E)^2/E  91.6 366.3 732.6 976.8 974.8 775.5 485.6 201.1  30.0  91.6 533681.1
       Birthday Spacings: Sum(O-E)^2/E=538407.147, p= 1.000
```

The second `tuftest` test is based on the number of steps to find the greatest common denominator by Euclid's (ca. 330–225BC) algorithm (the world's oldest surving nontrivial algorithm in mathematics), and on the expected distribution of the partial quotients.

```
func gcd(x,y) \
{
        rem = abs(x) % abs(y)
        if (rem == 0) return abs(y) else return gcd(y, rem)
}

proc gcdshow(x,y) \
{
        rem = abs(x) % abs(y)
        println x, "=", int(x/y), "*", y, "+", rem
        if (rem == 0) return
        gcdshow(y, rem)
}

gcd(366,297)
    3

gcdshow(366,297)
    366 = 1 * 297 + 69
    297 = 4 *  69 + 21
     69 = 3 *  21 +  6
     21 = 3 *   6 +  3
      6 = 2 *   3 +  0
```

This took $k = 5$ iterations, and found partial quotients $(1, 4, 3, 3, 2)$.

Interestingly, the complete rigorous analysis of the number of steps required in Euclid's algorithm was not achieved until 1970–1990! The average number is

$$\begin{aligned} A\left(\gcd(x,y)\right) &\approx \left((12\ln 2)/\pi^2\right)\ln y \\ &\approx 1.9405\log_{10} y \end{aligned}$$

and the maximum number is

$$\begin{aligned} M(\gcd(x,y)) &= \lfloor \log_\phi ((3-\phi)y) \rfloor \\ &\approx 4.785 \log_{10} y + 0.6723 \end{aligned}$$

where $\phi = (1+\sqrt{5})/2 \approx 1.6180$ is the golden ratio. For our example above, we find

$$\begin{aligned} A(\gcd(366,297)) &\approx 4.798 \\ M(\gcd(366,297)) &\approx 12.50 \end{aligned}$$

Here are two `tuftest` reports:

Good generator

```
Euclid's algorithm:
 p-value, steps to gcd:   0.452886
 p-value, dist. of gcd's: 0.751558
```

Bad generator

```
Euclid's algorithm:
 p-value, steps to gcd:   1.000000
 p-value, dist. of gcd's: 1.000000
```

The third `tuftest` test is a generalization of the monkey test: a monkey typing randomly produces a stream of characters, some of which eventually form words, sentences, paragraphs, ...

Good generator

```
Gorilla test for 2^26 bits, positions 0 to 31:
Note: lengthy test---for example, ~20 minutes for 850MHz PC
Bits  0 to  7---> 0.797 0.480 0.096 0.660 0.102 0.071 0.811 0.831
Bits  8 to 15---> 0.731 0.110 0.713 0.624 0.019 0.405 0.664 0.892
Bits 16 to 23---> 0.311 0.463 0.251 0.670 0.854 0.414 0.221 0.563
Bits 24 to 31---> 0.613 0.562 0.191 0.830 0.284 0.752 0.739 0.356
KS test for the above 32 p values:  0.289
```

Bad generator

```
Gorilla test for 2^26 bits, positions 0 to 31:
Note: lengthy test---for example, ~20 minutes for 850MHz PC
Bits  0 to  7---> 0.000 0.000 0.000 0.000 0.000 1.000 1.000 1.000
Bits  8 to 15---> 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000
Bits 16 to 23---> 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000
Bits 24 to 31---> 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000
KS test for the above 32 p values:  1.000
```

18 Further reading

The definitive work on computer generation of sequences of pseudo-random number is Chapter 3 of Donald E. Knuth, *The Art of Computer Programming, Vol. 2, Seminumerical Algorithms*, 3rd ed., Addison-Wesley (1998), ISBN 0-201-89684-2.

Douglas Lehmer and George Marsaglia have probably written more technical papers on the subject than anyone else: look for them with `bibsearch` and in the **MathSciNet** database.

Marsaglia's *Diehard Battery* test suite is available at:

`http://www.stat.fsu.edu/pub/diehard/`

Marsaglia and Tsang's `tuftest` package is described in *Some Difficult-to-pass Tests of Randomness*, J. Stat. Software **7**(1) 1–8 (2002):

`http://www.jstatsoft.org/v07/i03/tuftests.pdf`
`http://www.jstatsoft.org/v07/i03/tuftests.c`

For a history of the Central-Limit Theorem, see

`http://mathsrv.ku-eichstaett.de/MGF/homes/didmath/seite/1850.pdf`

For a real-time demonstration of the Central-Limit Theorem based on balls threading through a grid of pins, visit

`http://www.rand.org/methodology/stat/applets/clt.html`

For another live demonstration based on dice throws, visit

`http://www.math.csusb.edu/faculty/stanton/probstat/clt.html`

See Simon Singh's *The Code Book: the evolution of secrecy from Mary, Queen of Scots, to quantum cryptography*, Doubleday (1999), ISBN 0-385-49531-5, for a fine introduction to cryptography through the ages. Journals in the field are: *Cryptologia*, *Designs, Codes, and Cryptography*, and *Journal of Cryptology*.

For generation of truly-random sequences, see Peter Gutmann's book *Cryptographic Security Architecture: Design and Verification*, Springer-Verlag (2002) ISBN 0-387-95387-6. Chapter 6 of his Ph.D. thesis is available at

`http://www.cryptoengines.com/~peter/06_random.pdf`