First Midterm Due Monday, February 3 at 4:35.
Math 2270

1. Indicate whether each of the following statements is true or false. If true, prove it. If false, provide a counterexample.

(a) Each pair of diagonal $n \times n$ matrices commutes with each other.

(b) Each diagonal $n \times n$ matrix commutes with all $n \times n$ matrices.

(c) The product of two symmetric $n \times n$ matrices is symmetric.

(d) The product of two invertible $n \times n$ matrices is invertible.

(e) The product of two non-invertible $n \times n$ matrices is not invertible.

2. The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is the sum:

$$tr(A) = \sum_{i=1}^{n} a_{ii} \text{ of the diagonal entries}$$

(a) Show that the trace satisfies:

$$tr({}^tA) = tr(A)$$

for all $n \times n$ matrices.

(b) Show that the trace satisfies:

$$tr(AB) = tr(BA)$$

for all pairs of $n \times n$ matrices.

(c) Use (b) cleverly to show that the trace satisfies:

$$tr(ABA^{-1}) = tr(B)$$

for all pairs of matrices such that A is invertible.

Definition. If $W \subseteq V$ is a subspace of a given vector space V and $L : V \to W$ is a linear mapping, then L is called a **projection** if $L(\underline{w}) = \underline{w}$ for all vectors $\underline{w} \in W$.

3. Let V be the vector space of all $n \times n$ matrices over $\mathbb{R}$. Let $W \subset V$ be the subspace consisting of all symmetric matrices.

(a) Find a projection mapping from V to W.

(b) Describe the matrices that are in the kernel of the projection mapping that you found in (a).

(c) Compute the dimension of the kernel of the projection mapping from (a) directly, and check that this dimension plus the dimension of W is equal to the dimension of V.

4. For each of the following matrices A.

(i) Find a basis for the image of the associated mapping L_A.

(ii) Find a basis for the kernel of the associated linear mapping L_A.

(iii) If the matrix is invertible, find its inverse.

(a)

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$