Third (and Final) Midterm Due Monday, April 28
Math 2270

1. Let A be an operator on a finite-dimensional vector space V.

(a) Prove that A has a (nonzero, maybe complex) eigenvector.

Let B be another operator on V that commutes with A, i.e.

$$AB = BA$$

(b) Prove that A and B share a common (nonzero) eigenvector.

Let $A_1, \ldots, A_n$ be commuting operators on V.

(c) Prove that $A_1, \ldots, A_n$ share a common nonzero eigenvector.

Note. In general, these eigenvectors will not have the same eigenvalues.

2. Suppose A is an $n \times n$ matrix.

(a) Define the Jordan normal form for A.

(b) Sketch the proof that a suitable basis on $\mathbb{C}^n$ puts A into Jordan normal form.

3. Consider the differential operator:

$$D_\alpha = \left(\frac{d}{dx} - \alpha\right) \text{ for some real number } \alpha$$

This, for example, satisfies:

$$D_\alpha(\sin(x)) = \frac{d}{dx}\sin(x) - \alpha\sin(x) = \cos(x) - \alpha\sin(x)$$

(a) Find the one-dimensional kernel of the operator D_α.

(b) Find the n-dimensional kernel V of the operator $(D_\alpha)^n$.

(c) Show that D_α is an operator on the vector space V from (b).

(d) Find a cyclic vector in V for the operator D_α.

4. Find an invertible matrix B so that $B^{-1}AB$ is in Jordan normal form, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

What is the Jordan normal form?

5. For each of the following statements, either prove it or else give a counterexample.

(a) If all the eigenvalues of an $n \times n$ matrix are different from zero, the matrix is invertible.

(b) If an $n \times n$ matrix has zero as an eigenvalue, it is not invertible.

(c) The eigenvalues of a matrix with real entries are all real.

(d) Every symmetric $n \times n$ matrix has a real eigenbasis.

(e) Every real unitary $n \times n$ matrix has a real eigenbasis.

(f) Every $n \times n$ matrix has at most n eigenvalues.

(g) If $P(t)$ is the characteristic polynomial of A, then $P(A) = 0$.

6. Find a 5×5 matrix A with all of the following properties:

$$(A - 2I_5)^3(A - I_n)^2 = 0$$

but

$$(A - 2I_5)^2(A - I_n)^2 \neq 0 \text{ and } (A - 2I_5)^3(A - I_5) \neq 0$$