Math 4800

8/24

Math 4800

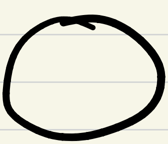
Category, Symmetry, Manifolds

How to represent symmetries as matrices?

- Finite symmetries
- Lie groups (physics)

Math we'll be looking at

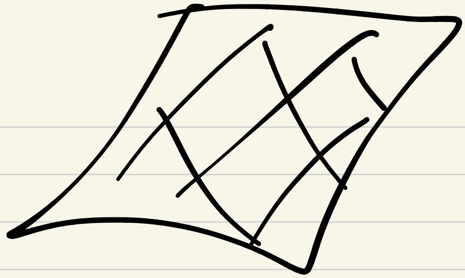
- Set theory
- Abelian groups
- Metric Spaces
- Vector Spaces
- Groups (e.g. permutations)
- Representing finite groups

S^1 

S^3 

$S^3 = SU(2)$

Math 2



Topology

$\mathbb{R}$ and $\mathbb{R}^n$ (Calc 1&2, Calc 3)

?! !?

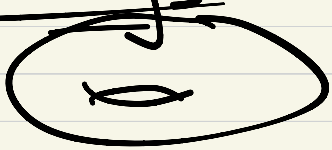
Manifold

(grad-level)

(idea)

Lie Groups

Flags



Webpage

www.math.utah.edu/~bertram

/4800

- Preview
- Sets + Exercises
- ; Notes

- Weekly assignments

+ Project

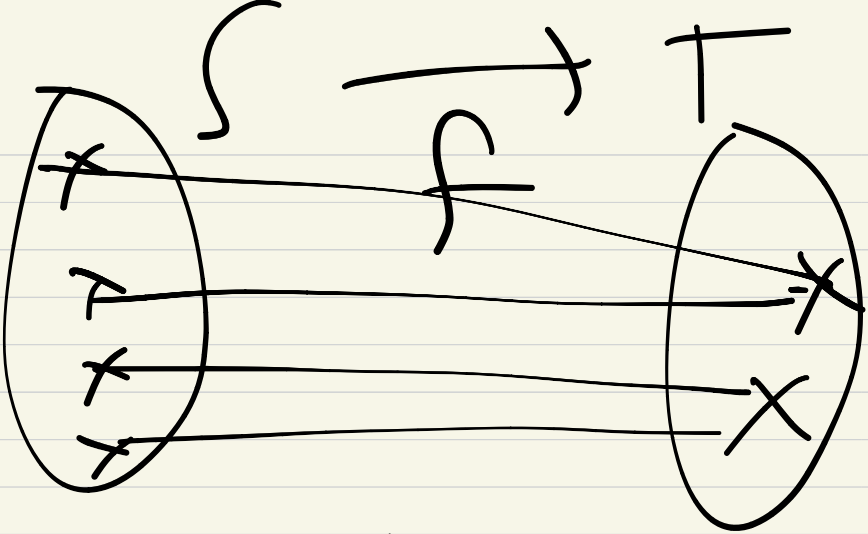
What's a category?

- Formal Definition. —
- Environment for mathematics problems.

Example: Sets

- Consist of elements
- Mappings/functions

$$f: S \to T$$

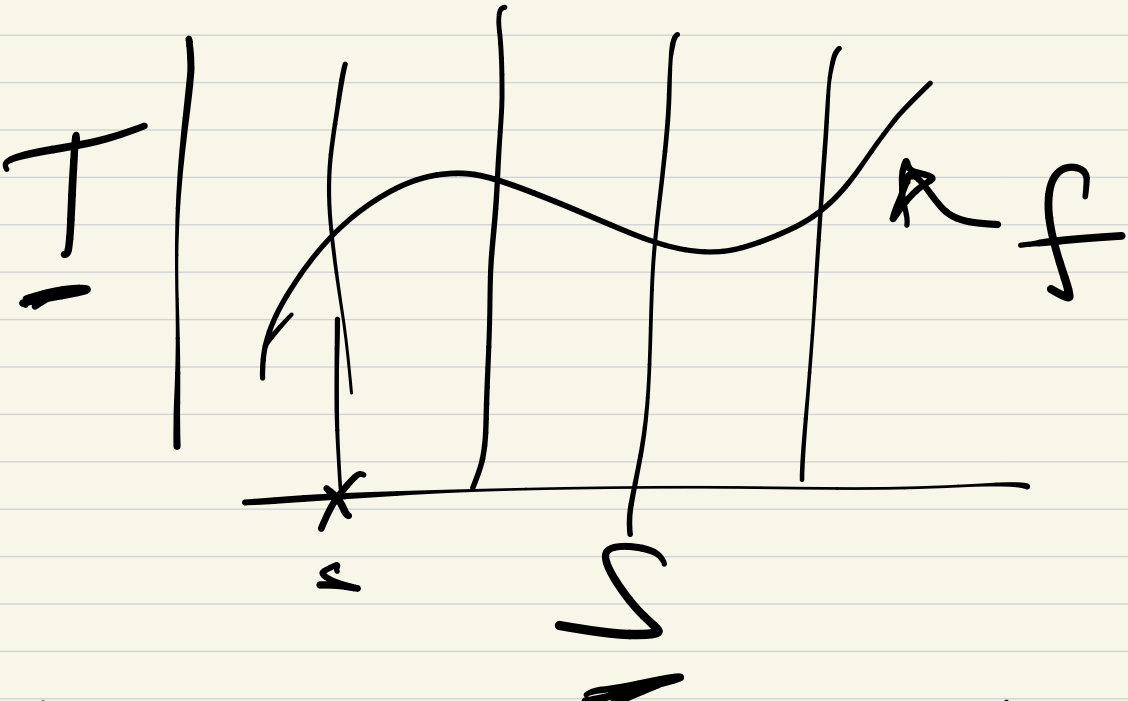


$f(s) = t$

What do we know about <u>functions</u> <u>of</u> <u>sets</u>?

Assign one element of T to each element of S.

A function can be thought of via its <u>graph</u> as a subset of $S \times T$



A function can be thought of as a <u>relation</u> - -

Functions Compose:

$$f : S \to T \qquad g : T \to U$$

$$g \circ f : S \to T$$

$$(g \circ f)(s) = g(f(s))$$

Composition is an operation on functions

$$(f, g) \longmapsto g \circ f$$

Composition is NOT comm.

is associative.

$(h \circ g) \circ f(s) =$

$h \circ (g \circ f)(s) = h(g(f(s)))$

There is always the id

function $1_S : S \to S$

$1_S(s) = s$.

A Category consists of:

- A collection of objects
- Sets of functions

$$\hom(S, T)$$

from S to T.

- Composition operation on functions that is associative.
- Identity functions 1_S

Examples:

- Sets + functions
- Vector spaces + linear maps
- Metric spaces + distance decreasing maps
- Groups + homomorphisms
- Topological spaces + continuous maps

⋮

- Manifolds + diff maps

In the context of a category, a <u>symmetry</u> is a function

$$f : X \longrightarrow X$$

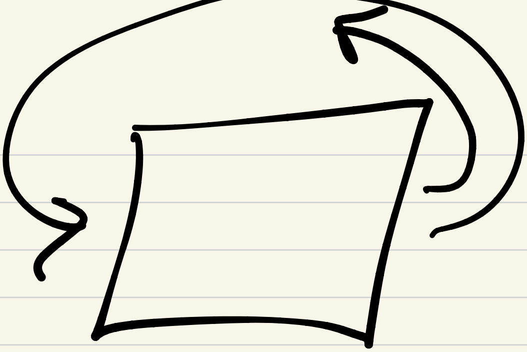
that has an <u>inverse</u>.

<u>Ex:</u> <u>Symmetry</u>

<u>Sets</u>: <u>Permutations</u>

<u>Vector spaces</u>: <u>Invertible</u> $n \times n$ <u>matrices</u>

<u>Metric spaces</u>: Isometries

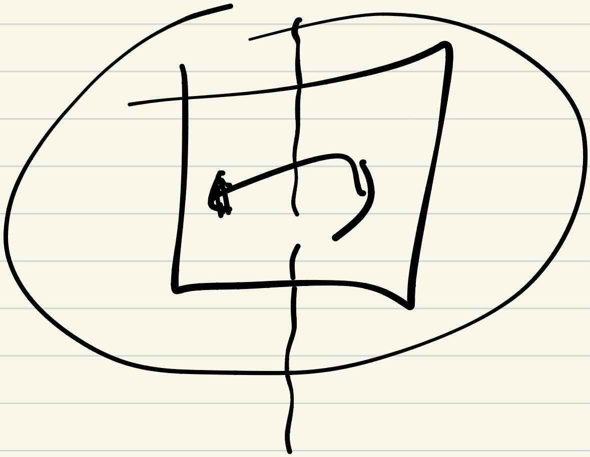


Symmetry

Isometries of the square

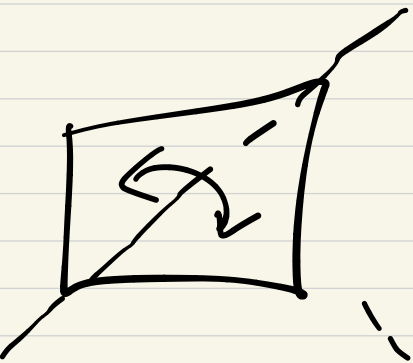
$\stackrel{?}{=}$ $1 = 360^\circ$

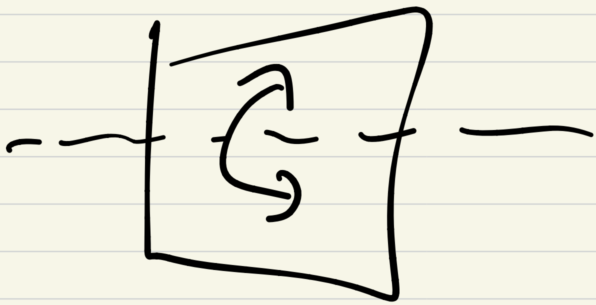
$\boxed{90^\circ} \longleftrightarrow$

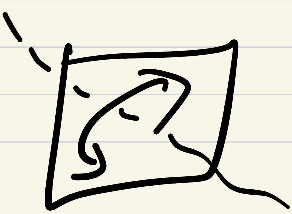


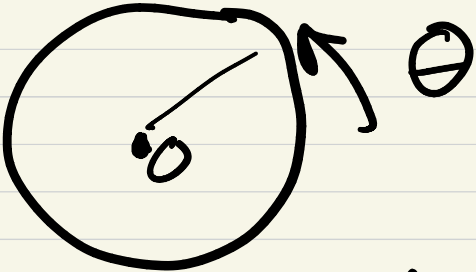
180°

270°



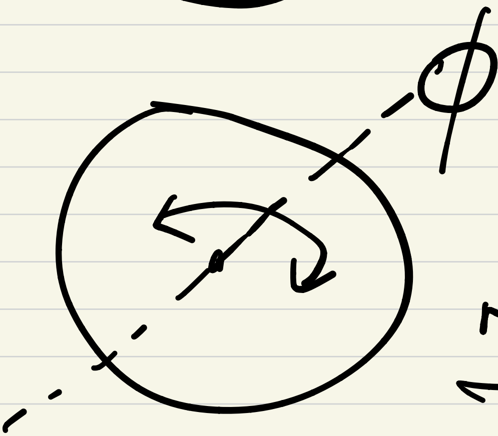




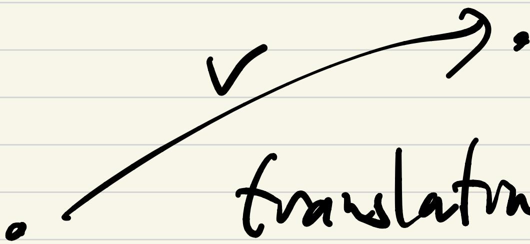


any angle

rotation



reflection



translation by v

· glides

⋮

Representation of a group

$$1 \qquad\qquad G$$

$$\parallel$$

$$\{1, -1 \mid -1^2 = 1\}$$

Two reps:

$$f : G \longrightarrow \text{Invertible } n \times n \text{ matrices}$$

$$f(g_1 \circ g_2) = f(g_1) \cdot f(g_2)$$

Permutation Matrices

ne

$$
\begin{matrix} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{matrix} \xrightarrow{f} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
$$

perm.

$\{1,2,3\}$

||Model||