4800-17

Given a group G,

a character of G is

a one-dim'l cx representation.

i.e. a group homomorphism

$$\chi : G \to \mathbb{C}^*$$

E.g. $\chi(g) = 1 \ \forall g$ is the

trivial character.

$\chi(\sigma) = \operatorname{sgn}(\sigma)$ is the

sign character of S_n.

Characters of C_n:

$$\chi_1, \dots \dots, \chi_n (= \chi_0)$$

$$\chi_m(\underset{\substack{\uparrow \\ \text{generator}}}{x}) = \zeta^m$$

where $\zeta = e^{2\pi i/n}$

$$\chi_m(x) = \zeta^m, \quad \chi_m(x^2) = \zeta^{2m}$$

$$\dots, \quad \chi_m(x^n) = (\zeta^m)^n = 1$$

Prop: If A is **abelian**, then every irreducible **complex** rep'n of A is a **character**.

$gh = hg$

Pf: If $\rho : A \to \mathrm{Aut}(\mathbb{C}^n)$ is a representation, then

$$\rho(g) = B_g \quad (n \times n \text{ matrix})$$

and

$$\rho(h) = B_h$$

commute.

So $\rho \rightsquigarrow B_g$ commuting matrices

[**Fact**: Any set of commuting matrices shares an eigenvector.]

($\Longrightarrow$ Let v be such an eigenvector. Then

$\langle v \rangle = \mathbb{C} \cdot v$ the span of v

is invariant in $\mathbb{C}^n$)
A

Start with B_g and let v be an eigenvector with eigenvalue λ.

$$B_g v = \lambda v$$

Suppose $B_h B_g = B_g B_h$. Then

$$B_g(\overbrace{B_h v}^{``w"}) = B_h(B_g v) = B_h \lambda v$$

$$= \lambda(\overbrace{B_h v}^{``w"})$$

$$B_g v = \lambda v$$

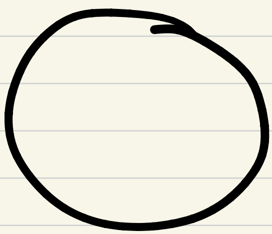
$$B_g(B_h v) = \lambda(B_h v)$$

<u>Consider</u>:

$$B_h : \begin{pmatrix} \overset{w}{\lambda}\text{-eigenspace} \\ \text{of } B_g \end{pmatrix} \circlearrowleft$$

Let w be an eigenvector for B_h (eigenvalue μ) <u>inside</u> the λ-eigenspace of B_g.

This shows that any <u>finite</u> set of commuting matrices has a common eigenvector. $\square$



S^1

E.g

Take the "cyclic" repn of C_n: on $\mathbb{C}^n$

$$\rho_{cyc}(x)(e_i) = e_{i+1} \qquad \underline{i < n}$$

$$\rho_{cyc}(x)(e_n) = e_1$$

$$\rho_{cyc}(x) = \begin{bmatrix} 0 & 0 & & 0 & 1 \\ 1 & 0 & & \vdots & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots \\ \vdots & \vdots & & 1 & 0 \end{bmatrix}$$

The following are eigenvectors for $\rho_{cyc}(x)$:

	value
$e_1 + e_2 + \cdots + e_n$	1
$e_1 + \zeta e_2 + \zeta^2 e_3 + \cdots + \zeta^{n-1} e_n$	ζ^{-1}
$e_1 + \zeta^2 e_2 + \zeta^4 e_3 + \cdots$	ζ^{-2}
$\vdots$	ζ^1

$$\zeta^{-1}(e_1 + \zeta e_2 + \zeta^2 e_3 + \cdots)$$

$$= e_2 + \zeta e_3 + \cdots + \zeta^{-1} e_1 = \rho(x)(\quad)$$

Example:

abelian gp

$$\rho : (\mathbb{C}, +, 0) \xrightarrow{\text{cx rep}} \mathrm{Aut}(\mathbb{C}^2)$$

$$\rho(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$$

$$\rho(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\rho(z+w) = \begin{bmatrix} 1 & z+w \\ 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}}_{\rho(z)} \cdot \underbrace{\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}}_{\rho(w)}$$

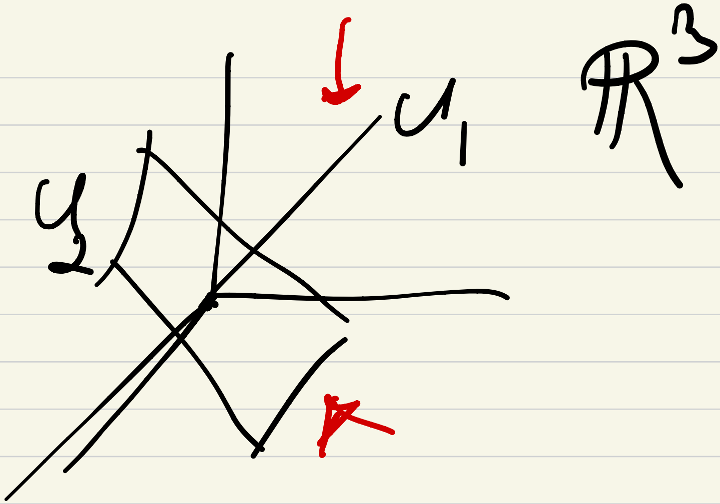
e_1 is "the" common eigenvector of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Def: (i) If $U_1, \dots, U_m \subset V$ are subspaces, then

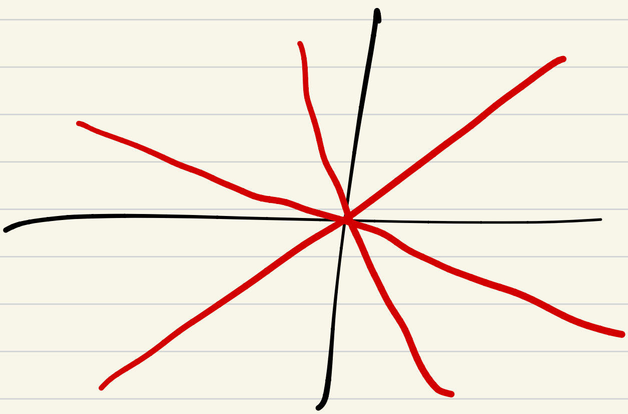
$$V = U_1 \oplus \dots \oplus U_m \text{ if } \forall v \in V$$

$$\exists! \ v = u_1 + \dots + u_m \text{ s.t. } u_i \in U_i.$$

Ex:



$$\mathbb{R}^3 = U_1 \oplus U_2$$



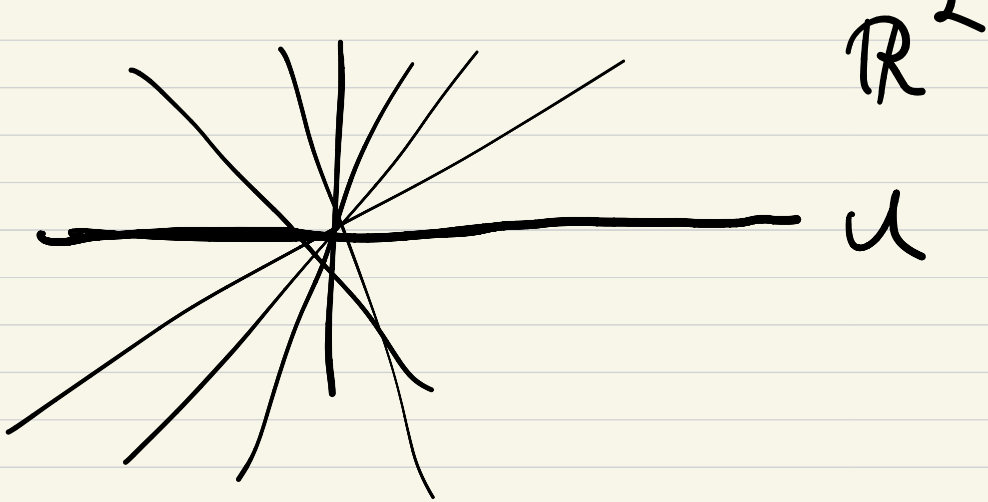
Not a direct sum!

Given $U \subset V$, then

a subspace $W \subset V$ is

called a <u>complement</u> of

U if $U \oplus W = V$



Complements of subspaces abound.

(ii) Given (V, ρ) a representation and a G-invariant $U \subset V$, then

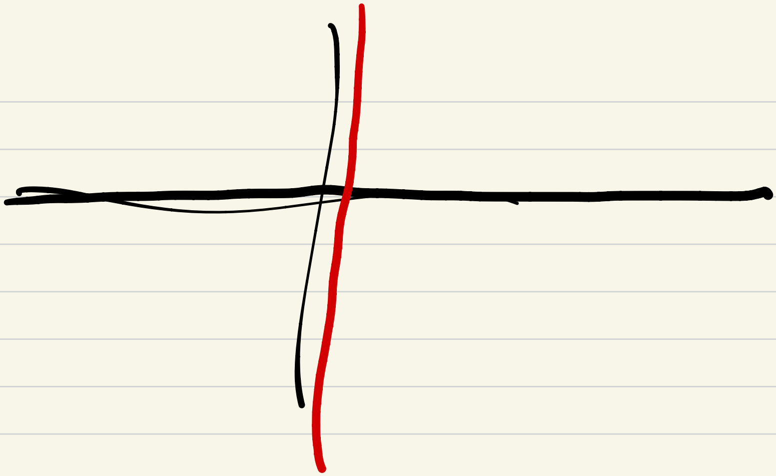
$V = U \oplus W$ as a $\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]$ representation of W is also G-invariant.

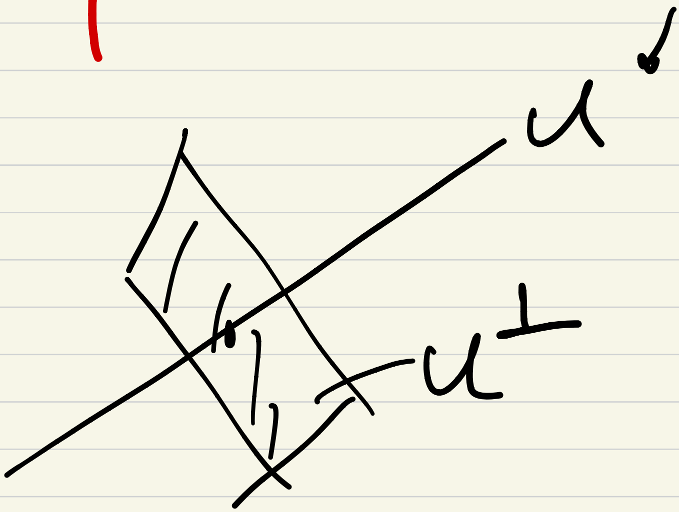
<u>Those are rare</u>: $\rho(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}$

$\to U = \langle e_1 \rangle$ has no invariant comp!

Prop: If G is finite,
then every G-invariant
subspace $U \subset V$ (for (V, ρ))
has an invariant complement!

Idea: Recall that if
$U \subset \mathbb{R}^n$ or $U \subset \mathbb{C}^n$, then
U has the orthogonal complement
$U^\perp = \{v \mid u \cdot v = 0\}$; $U^\perp = \{v \mid \langle u, v \rangle = 0\}$





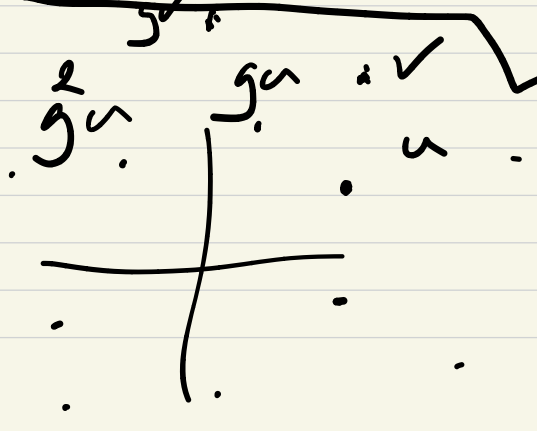
Idea: To modify the Hermitian inner product so that $U^{\perp}$ is G-invariant.

Given (V, ρ) complex representation of G.
on $\mathbb{C}^n$

Create a new Hermitian inner product by averaging(!)

$u, v \in \mathbb{C}^n$

$$\boxed{\langle u, v\rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle gu, gv\rangle}$$



Notice that

$$\boxed{\langle u, v\rangle_G = \langle h\cdot u, hv\rangle_G}$$

$$\langle u, v\rangle_G = \boxed{\frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle}$$

$$\langle h\cdot u, hv\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle ghu, ghv\rangle = \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle$$

$$\boxed{\langle u, u\rangle = \frac{1}{|G|}\sum |gu|^2 > 0}$$

Rmk: Given $U \subset V$ invariant, then

$$\boxed{U^{\perp} = \{ v \in V \mid \langle u, v \rangle_G = 0 \}}$$

is also invariant!

$\langle u, v \rangle_G = 0$ $\langle u, gv \rangle_G$

$\forall u$

$= \langle g^{-1}u, g^{-1}gv \rangle_G$

$= \langle g^{-1}u, v \rangle = 0$

Run this on a rep.

of $C_2 = \{\pm 1\}$.

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(-1) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$U = \langle e_1 \rangle$

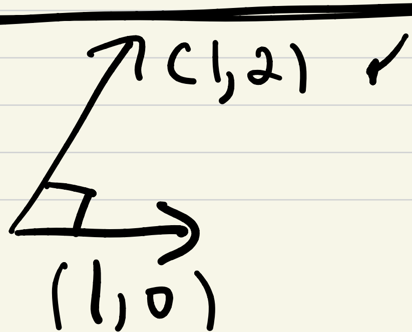
$$\langle e_1, e_1 \rangle_G = \frac{1}{2}\left(\langle e_1, e_1 \rangle + \langle e_1, -e_1 \rangle\right) = 1$$

$$\langle e_1, e_2 \rangle_G = \frac{1}{2}\left(\underset{0}{\langle e_1, e_2 \rangle} + \underset{-1}{\langle -e_1, e_1 + e_2 \rangle}\right) = -\frac{1}{2}$$

$$\langle e_1, \underline{e_1 + 2e_2} \rangle_G = 1 + 2\left(-\tfrac{1}{2}\right) = 0$$

<u>Claim</u>: $\langle e_1 + 2e_2 \rangle$ is <u>also</u> invariant. In fact,

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$S_3 \text{ on } \mathbb{C}^3$$

$$\circlearrowright$$

$$\langle e_1+e_2+e_3\rangle \quad \langle e_1+e_2+e_3\rangle^{\perp}$$

$$= \underline{\underline{\langle e_1-e_2, e_2-e_3\rangle}}$$

__Cor__: If $A \in Aut(\mathbb{C}^n)$ and $\underline{\underline{A^d = I_n}}$, then A is semi-simple!

__Pf__: Think of the representation of C_d given by $\underline{\underline{\rho(x) = A}}$.

Then A has an eigenvector,

i.e. an invariant subspace

$U = \langle v \rangle \subset A$

Take $U^{\perp}$ for $\langle -, \cdot \rangle_{\mathbb{C}^n}$

Then $\underline{A : U^{\perp} \to U^{\perp}}$

so this trans. is semisimple,

$\Rightarrow$ A has a basis of

eigenvectors.