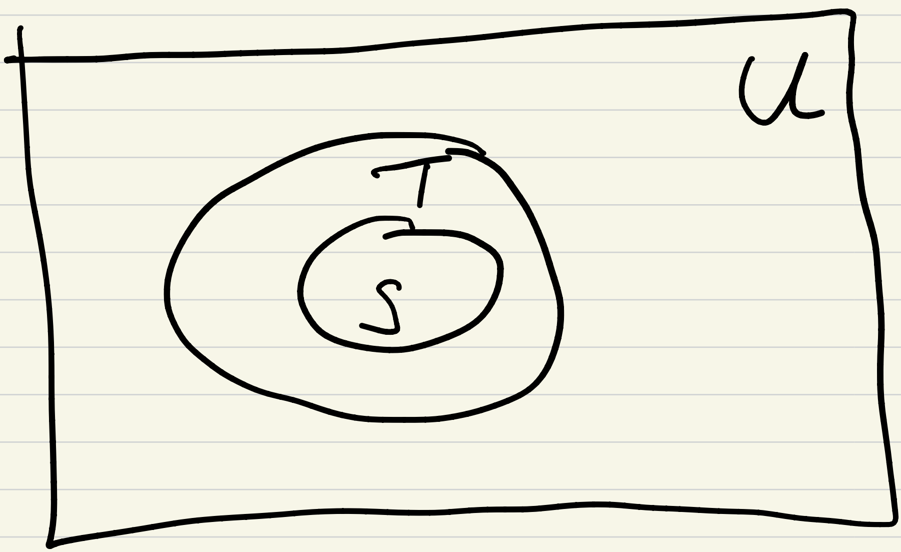
<u>Categories</u> <u>of</u> <u>Sets</u>

(<u>Tale of Two Categories</u>)

Subsets of $\mathcal{U}$

"<u>universe</u>"



$\mathcal{U} = \mathbb{R}$, <u>any</u> <u>set</u>

$\mathrm{Sub}_{\mathcal{U}}$ category of subsets of $\mathcal{U}$

- Objects subsets of $\mathcal{U}$

$$S, T, \ldots \subseteq \mathcal{U}$$

- ~~Functions~~ Morphisms set inclusions

$$S \subseteq T$$

Note: $\mathrm{hom}(S,T) = \begin{cases} \{\subseteq\} \\ \text{or} \\ \emptyset \end{cases}$

$R \subseteq S \subseteq T$ and $S = S$

$\underline{\mathcal{U} = \emptyset}$

<u>one</u> <u>object</u>: $\emptyset$

<u>one</u> <u>morphism</u>: $\emptyset = \emptyset$

$$\begin{tikzcd} \emptyset \arrow[loop above, "="] \end{tikzcd}$$

$\mathcal{U} = \{0\}$

<u>two</u> <u>objects</u> $\emptyset$ $\{0\}$

$\emptyset \subseteq \{0\}$ $\qquad$ $\{0\} \subseteq \{0\}$

$$\begin{tikzcd} \emptyset \arrow[loop above, "="] \arrow[r, "\subseteq"] & \{0\} \arrow[loop above, "="] \end{tikzcd}$$

$\mathcal{U} = \{0,1\}$

· $\underline{4}$ <u>objects</u>

$$\boxed{\begin{array}{c}\underline{\underline{\text{Subsets}}} \\ \cancel{\text{Symm}}\ \text{Do} = \ =\end{array}}$$

$\emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$

$$\begin{tikzcd}
 & \{0\} \arrow[loop above, "="] \arrow[dr] & \\
\emptyset \arrow[loop above, "="] \arrow[ur] \arrow[rr] \arrow[dr] & & \{0,1\} \arrow[loop above, "="] \\
 & \{1\} \arrow[loop above, "="] \arrow[ur] &
\end{tikzcd}$$

initial object

final object

Three ~~Two~~ operations on objects

$\cap$, $\cup$, ^{c}

intersection union complement

Rmk: $\mathrm{Sub}_{\mathcal{U}}$

- The collection of objects of $\mathrm{Sub}_{\mathcal{U}}$ is a set

$\mathcal{P}(\mathcal{U})$ power set of $\mathcal{U}$.

Sets

- objects are sets,

$$S, T, U, \ldots$$

- morphisms are functions

$$f : S \to T$$

- $\hom(S, T) = \{S \to T\}$

$(=) = 1_S : S \longrightarrow S$

Russell's Paradox:

The collection of sets is **not** a set.

If $\mathcal{U}$ were the set of sets, then

- $\mathcal{U} \in \mathcal{U}$
- $X = \{ S \in \mathcal{U} \mid S \notin S \} \subseteq \mathcal{U}$

$X \in X$? **No** $\quad$ $X \notin X$? **No**

Picture of

Endomorphisms

$$4 = |\hom(\{1,2\},\{1,2\})|$$

=

not symm. -1 symm.

$f(1)=1$	$f(1)=1$	$f(1)=2$	$f(1)=2$
$f(2)=2$	$f(2)=1$	$f(2)=2$	$f(2)=1$

=

Def: A symmetry of X (object of a category) is an isomorphism

$$\begin{tikzcd} X \arrow[r, "f"] & X \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

In Sets, the permutations are the symmetries of finite sets.

$\boxed{\text{Paul Erdős}}$

Definitions:

$$[n] = \{1, \dots, n\}$$

$$[1] = \{1\}$$

$$[2] = \{1, 2\}$$

etc.

Every finite set is isomorphic with a <u>unique</u> $[n]$.

$$\exists\, S \hookrightarrow \mathbb{N}, \quad \nexists\, S \cancel{\twoheadrightarrow} \mathbb{N}$$

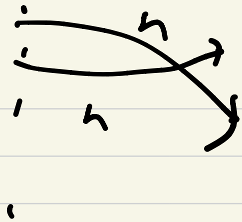
Counting Problems

① If $m<n$, count all the injective functions

$$f:[m] \longrightarrow [n]$$

$$\left(f \text{ is injective if } x \neq y \Longrightarrow f(x) \neq f(y) \right)$$

$$\left(f \text{ is surjective if } \forall x,\ f^{-1}(x) \neq \emptyset \right)$$



Answer to (1)

- n^m functions from $[m]$ to $[n]$
- $n(n-1)\cdots(n-m+1)$ injective. Easy!

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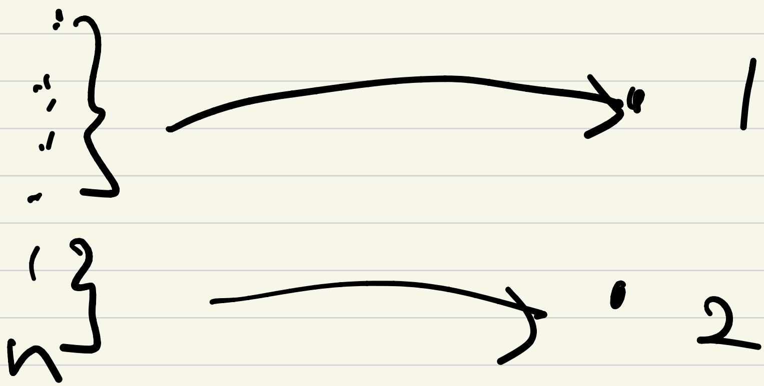
(2) Count the surjective functions from $[n]$ to $[m]$.

Hard!



The surjective maps

$\longleftrightarrow$ subsets of $[n]$
except for $\underline{\emptyset}$, $\underline{[n]}$

$$2^n - 2$$

<u>Try</u>: $\underline{m=3}$!!

Two Operations on Sets

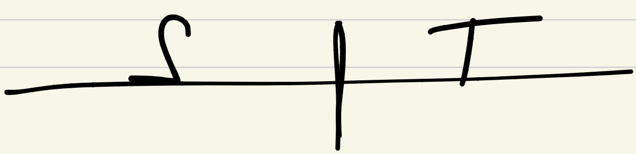
Given S, T, then

$$S \times T = \{ (s,t) \mid s \in S \text{ and } t \in T \}$$

Cartesian Product

$S \overset{+}{\cup} T$ disjoint union

is a set ~~that~~



$$S \sqcup T = \{(s,1) \mid s \in S\}$$
$$\cup \{(t,-1) \mid t \in T\}$$

<u>Want</u>: ~~If~~ S, T are <u>finite</u>

$$|S \times T| = |S| \cdot |T| .$$

&

$$|S \sqcup T| = |S| + |T| .$$

$$\big(|S \cup T| = |S| + |T| - |S \cap T|$$

$$(S \cup S \neq S).$$

Study symmetries of finite sets

Suffice to study symmetries of $[n]$.

Def. An isomorphism

$f(1) = s_1, f(2) = s_2, \ldots$

$f: [n] \longrightarrow S$ is

called an ordering of S

Transferring symmetries of $[n]$ to symmetries of S

σ = symmetry of $[n]$

$$\begin{tikzcd} {[n]} \arrow[loop left, "\sigma"] \arrow[r, shift left, "f"] & S \arrow[l, shift left, "f^{-1}"] \arrow[loop right, "f \circ \sigma \circ f^{-1}"] \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[loop left, "\sigma"] \arrow[r, shift left, "f"] & {} \arrow[l, shift left, "f^{-1}"] \arrow[loop right, "f \circ \sigma \circ f^{-1}"] \end{tikzcd}$$

$[n] = 2$

$\sigma(1) = 2$

$\sigma(2) = 1$

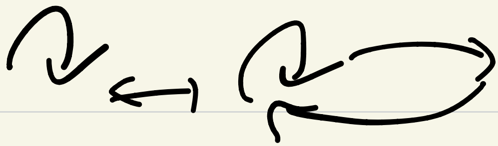
$$\{1, 2\} \xrightarrow{f} \{x, y\}$$

$f(1) = x$

$f(2) = y$

$$1 \leftrightarrow 2 \qquad x \leftrightarrow y$$

$$\overbrace{f \circ \underbrace{\sigma \circ \underbrace{f^{-1}(x)}_{1}}_{2}}^{\tau} = y$$



$\sigma_1 \circ \sigma_2$

$$\left(f\circ\sigma_1\circ \cancel{f^{-1}}\right)\left(\cancel{f}\circ\sigma_2\circ f^{-1}\right)$$

$$= f\circ\sigma_1\circ\sigma_2\circ f^{-1}$$

composition of transfers ‖
transfer of compositions !

$$\underline{\left(f\circ\sigma_1\circ f^{-1}\right)^{-1}} = \underline{f\circ\sigma_1^{-1}\circ f^{-1}}$$

Understand symmetries of $[n]$

Permutations of $[n]$

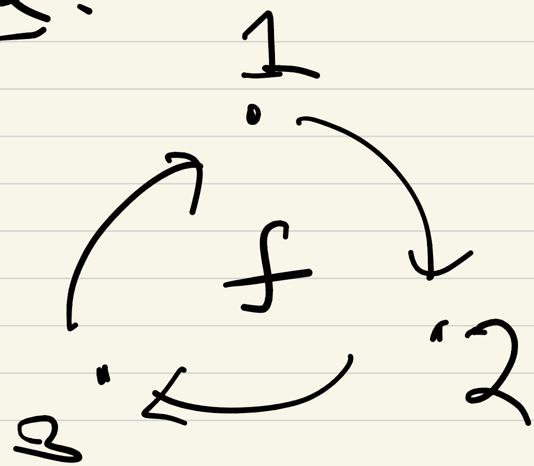
have a sign.

Let $f : [n] \to [n]$

Definition:

$$\mathrm{sgn}(f) = \prod_{i<j} \frac{f(j) - f(i)}{j - i}$$

Example:



$$f(1)=2,\ f(2)=3,\ f(3)=1$$

$$\operatorname{sgn}(f) = \prod_{i<j} \frac{f(j)-f(i)}{j-i}$$

$$= \underset{1<2}{\frac{\overset{3}{f(2)}-\overset{2}{f(1)}}{2-1}} \cdot \underset{1<3}{\frac{\overset{1}{f(3)}-\overset{2}{f(1)}}{3-1}} \cdot \underset{2<3}{\frac{\overset{1}{f(3)}-\overset{3}{f(2)}}{3-2}}$$

$$= 1 \cdot -\frac{1}{2} \cdot -2 = \underline{\underline{+1}}$$

Proposition:

(a) If f is not a symmetry, then $\operatorname{sgn}(f) = 0$

(b) otherwise $\operatorname{sgn}(f) = +1$ or -1

(c) $\operatorname{sgn}(f \circ g) = \operatorname{sgn}(f) \cdot \operatorname{sgn}(g)$