<u>4800-3</u>

<u>HW</u> open until <u>Monday</u>.

<u>Permutations</u>:

σ

$$[n] = \{1, \dots, n\}$$

Every permutation of $[n]$ has a <u>sign</u>

$$\underline{\operatorname{sgn}(\sigma)} = \prod_{i<j} \frac{\sigma(j) - \sigma(i)}{j - i}$$

$$= \frac{\prod(\sigma(j) - \sigma(i))}{\prod(j - i)}$$

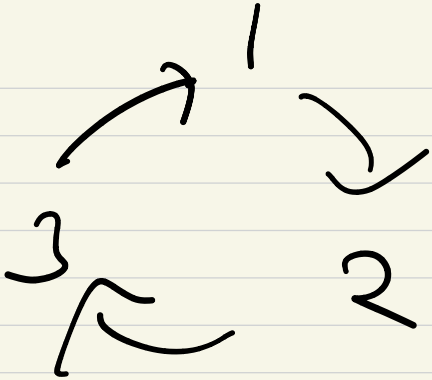
<u>Rmk:</u> As we range over all pairs $\{i, j\}$, the pairs $\{\sigma(i), \sigma(j)\}$ also range over all pairs (but maybe in "wrong" order)

$$\Rightarrow \prod |j - i| = \prod |\sigma(j) - \sigma(i)|$$

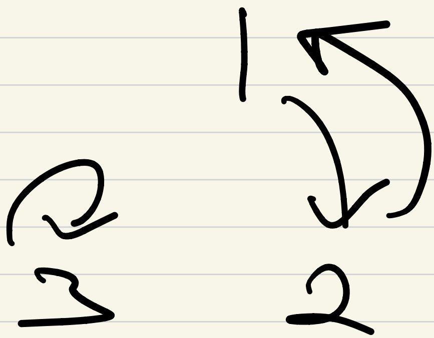
$$\prod \underbrace{|j-i|}_{\substack{\text{"} \\ |i-j|}} = \prod \underbrace{|\sigma(j)-\sigma(i)|}_{\substack{\text{"} \\ |\sigma(i)-\sigma(j)|}}$$

$$\Rightarrow \operatorname{sgn}(\sigma) = \pm 1$$

Example:



$+1$



$$\left(\frac{\overset{-}{1-2}}{\underset{+}{2-1}}\right)\cdot\left(\frac{\overset{+}{3-2}}{\underset{+}{3-1}}\right)\cdot\left(\frac{\overset{+}{3-1}}{\underset{+}{3-2}}\right) = -1$$

$\sigma(1)=2$
$\sigma(2)=1$
$\sigma(3)=3$

Property (composition $\to$ product)

σ, τ permutations

$\Rightarrow \operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau)$

Pf:

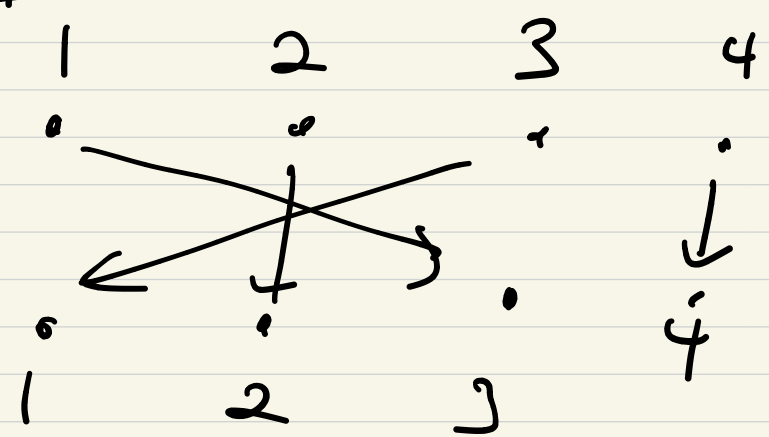
$$\operatorname{sgn}(\sigma \circ \tau) = \prod \frac{\sigma(\tau(j)) - \sigma(\tau(i))}{j - i}$$

$$= \prod \frac{\sigma(\tau(j)) - \sigma(\tau(i))}{\tau(j) - \tau(i)} \cdot \frac{\tau(j) - \tau(i)}{j - i}$$

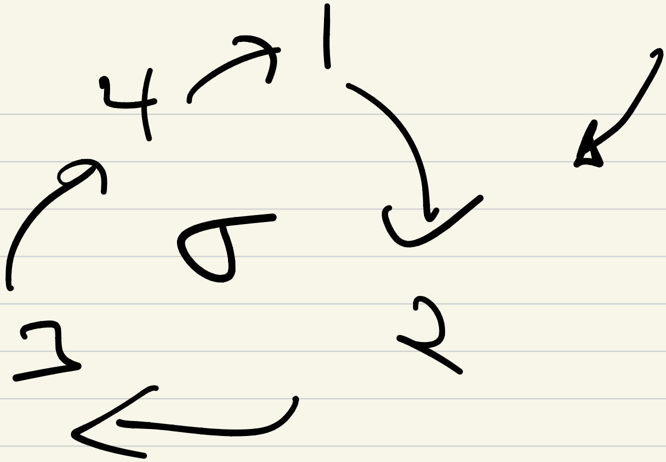
$\operatorname{sgn}(\sigma)$ $\operatorname{sgn}(\tau)$

Moral: If we can find permutations that "generate" all symmetries of $[n]$, their signs determine all the signs!

Transpositions



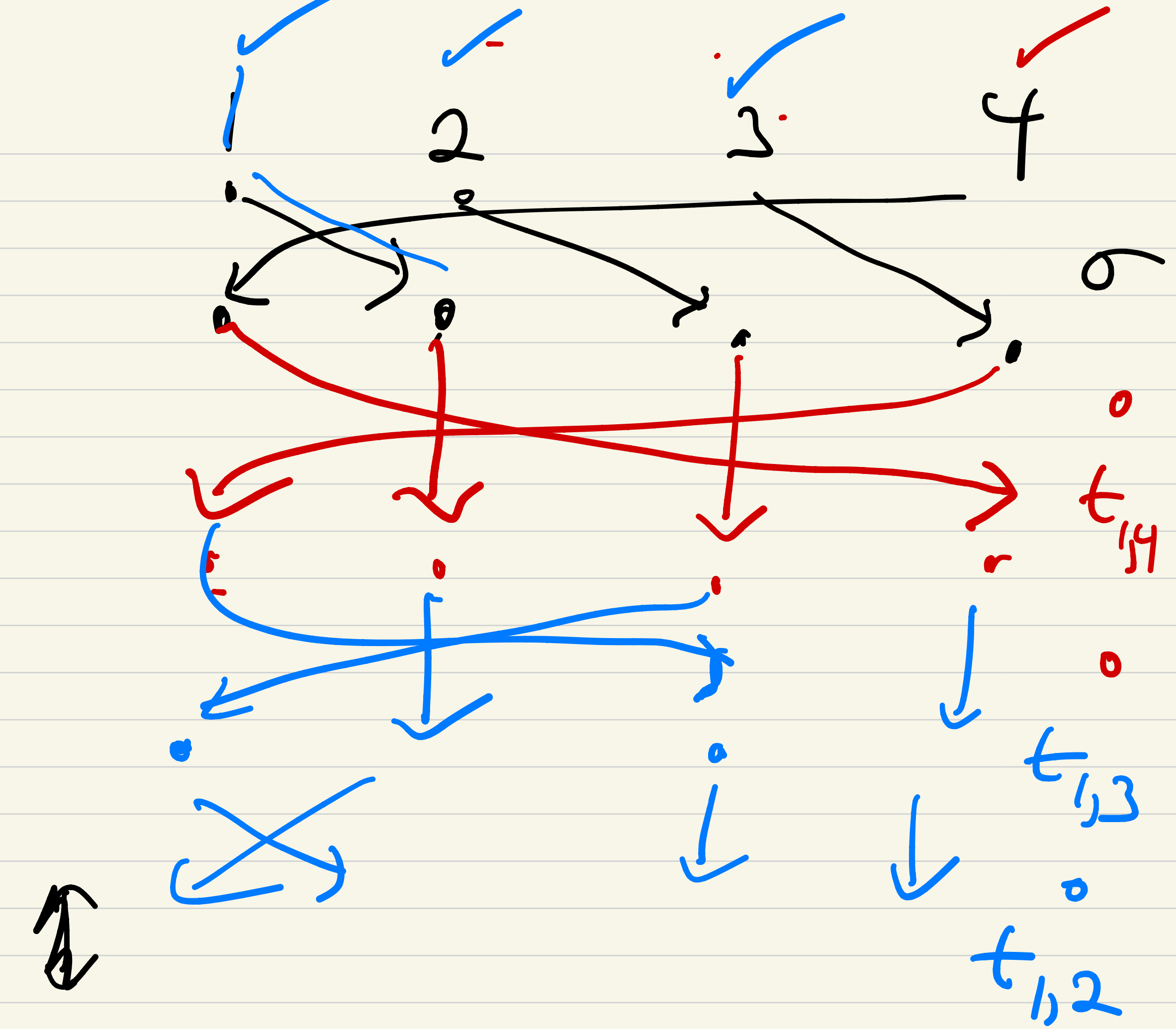
$\sigma(4) = 1$



Name: t_{ij} = transposition swapping i and j.

$$t_{1,4} \circ \sigma\,(4)$$

$$= t_{1,4}(1) = 4$$



$$t_{1,2} \circ t_{1,3} \circ t_{1,4} \circ \sigma = id$$

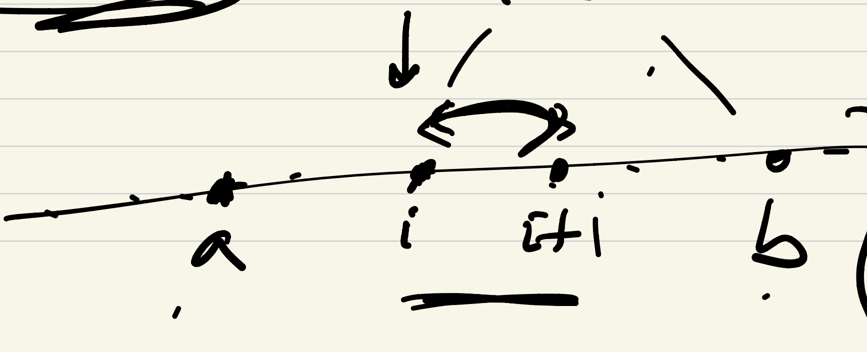
$$\cancel{t_{1,2} \circ t_{1,2}}\ t_{1,3} \circ t_{1,4} \circ \sigma = t_{1,2}$$

$$\cancel{t_{1,3} \circ t_{1,3}} \circ \cancel{t_{1,4}} \circ \boxed{\sigma = t_{1,4} \circ t_{1,3} \circ t_{1,2}}$$

Proposition: The transpositions $t_{12}, t_{23}, \dots, t_{n-1\,n}$ are enough to generate all symmetries

$$\text{sgn}(t_{i\,i+1}) = -1 \qquad \text{all} \quad +$$

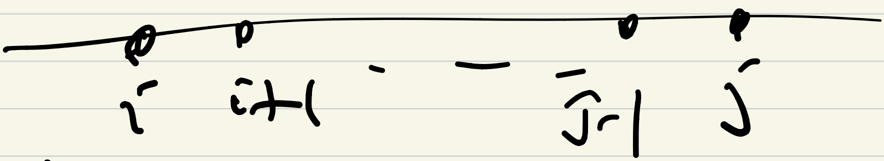
$$-1 = \text{sgn}(t_{i\,i+1}) = \prod_{a<b} \left(\frac{t(b)-t(a)}{b-a} \right)$$



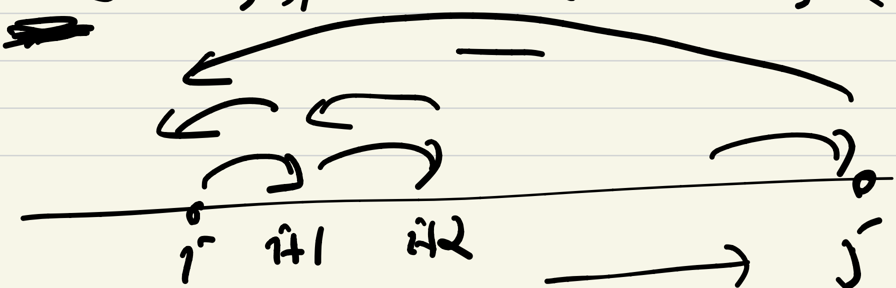
except for $i<i+1$

In fact, $\operatorname{sgn}(t_{ij}) = -1$

E.g. See this by expressing t_{ij} as a product:



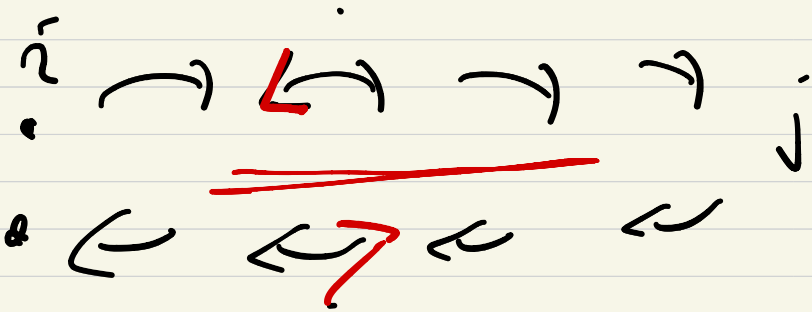
$$t_{ij} = t_{i,i+1} \circ \cdots \circ t_{j-1\,j} \circ \cdots \circ t_{i+1,i+2} \circ t_{i,i+1}$$



$$t = t_{1,2} \circ t_{23} \circ t_{34} \circ t_{23} \circ t_{12}$$

$t(1) = 4$

$t(2) = 2$



Upshot: $\operatorname{sgn}(t_{ij}) = -1$

Remarkable consequence:

Every permutation is a ~~product~~ composition of either an odd or even # of transpositions

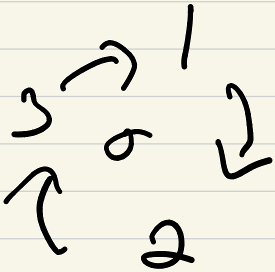
$$\text{odd} \Leftrightarrow (\mathrm{sgn}(\sigma) = -1)$$
$$\text{even} \Leftrightarrow (\mathrm{sgn}(\sigma) = +1)$$

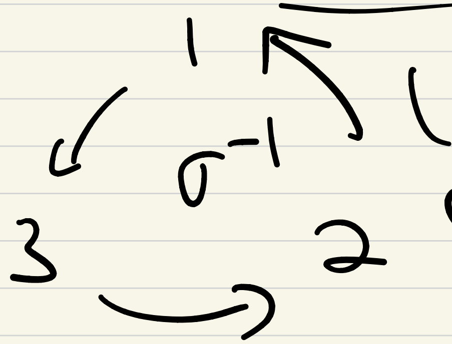
All permutations of $[3]$

$Id = 1_{[3]}$ $\quad$ <u>even</u>

$\underline{t_{12}}, \underline{t_{13}}, \underline{t_{23}}$ $\quad$ $\underline{\underline{\text{odd}}}$

$\sigma = \underline{t_{13}} \circ \underline{t_{12}}$ $\qquad$ $\underline{\sigma}^{-1} = \underline{t_{12} \circ t_{13}}$





$\underline{\underline{\text{even}}}$

$$\sigma = t_{13} \cdot t_{12} \qquad \begin{aligned} \sigma(1) &= 2 \\ (2) &= 3 \\ (3) &= 1 \end{aligned}$$

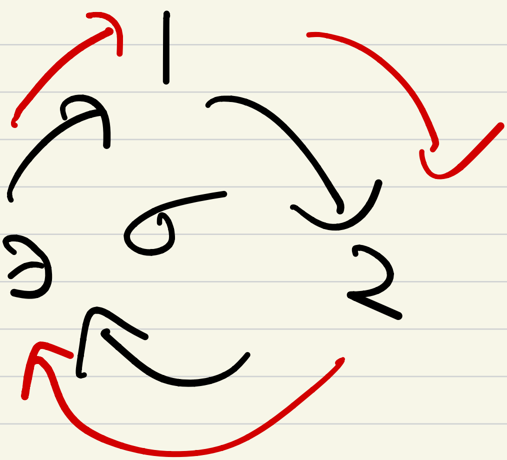
Fact: Exactly **half** the permutations of $[n]$ are **even**.

Fact: The even permutations are closed under **composition**.

The **even permutations** of $[n]$ are called the **alternating group**.

E.g. [3]

$$\left[\begin{array}{l} \sigma, \sigma^{-1}, id = \text{alt. gp.} \\ \sigma \circ \sigma = \sigma^{-1} \end{array}\right]$$



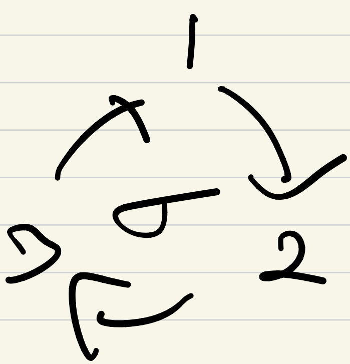
$$\sigma^2(1) = 3$$

$$\sigma^2(2) = 1$$

$$\sigma^2(3) = 2$$

Q: What's the best way to record a permutation?

A: (Cayley 1800's) via the orbits.



$(1\ 2\ 3) = \sigma$

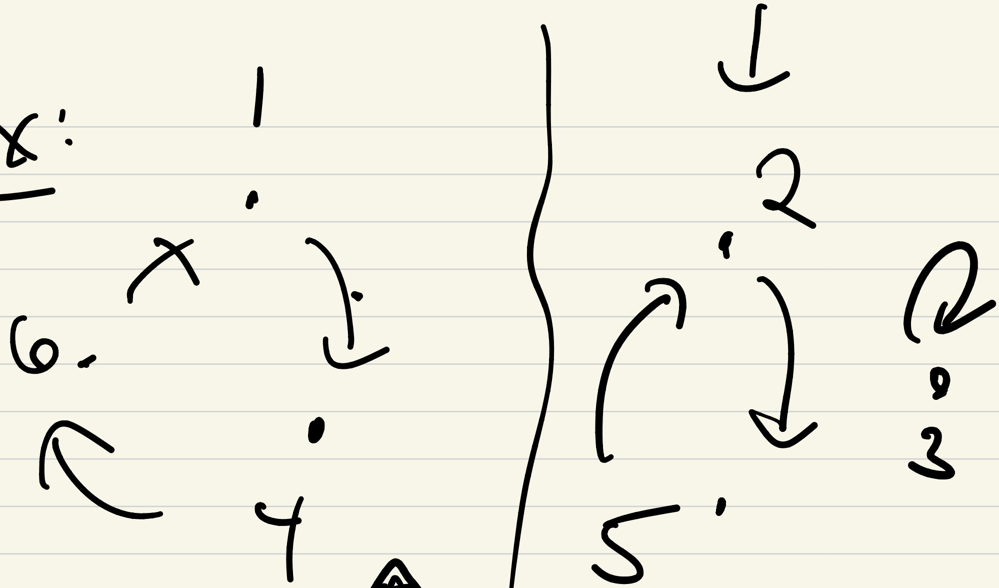
(1 travels) (missing)

a

from 1st orbit

Ex:

σ:



$$(1\ 4\ 6)(2\ 5)(3)$$

$\boxed{24}$

All permutations of [4]:

Q: what could the orbits looks like, qualitatively?

id: $(1)^1(2)^1(3)^1(4)^1$ even 1

tr: $(\quad)^2(\quad)^1(\quad)^1(\quad)^1$ odd $\binom{4}{2}=6$

2 cycle

two pair $(\quad)^2(\quad)^2$ even 3

3 cycle

3 of a kind $(\quad)^3(\quad)^1$ even 8

4cycle

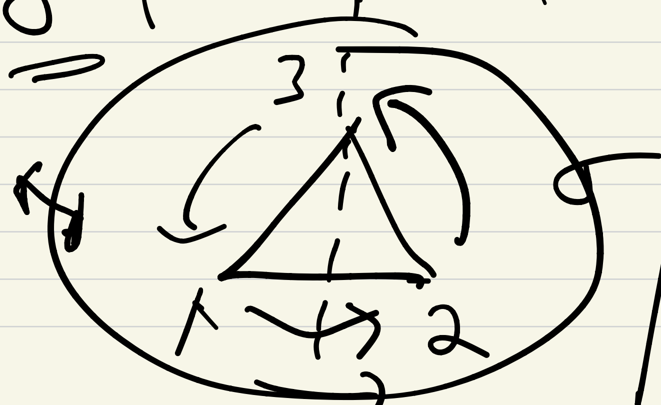
4 of a kind $(1 * * *)^4$ odd 6

$1 \to 2$
$2 \to 1 \to 3$
$3 \to 1$

$t_{13} \circ t_{12}$

Composition Table for [3]

$\circ$	(12) t_{12}	(13) t_{13}	(23) t_{23}	(123) σ	(132) σ^2
t_{12}	1	σ			
t_{13}	σ^2	1			
t_{23}			1		
σ				σ^2	1
σ^2				1	



fill in!

Symmetries of △!

Let $\mathcal{C}$ be a <u>category</u>

- objects
- functions/morphisms.

Let X, Y be objects.

The a <u>product</u> of X & Y is a third <u>object</u> Z together with ~~p~~ functions:

$$\left(\underline{Z \xrightarrow{p} X},\ \underline{Z \xrightarrow{f} Y}\right)$$

This triple should be "universal":

$$\begin{tikzcd}
W \arrow[r, dashed] \arrow[d, "f"'] \arrow[rrd, bend left, "g"] & Z \arrow[ld, "p"'] \arrow[rd, "q"] & \\
X & & Y
\end{tikzcd}$$

Examples: $X \times Y = \{(x,y)\}$ is the product of sets.

$$\begin{tikzcd}
W \arrow[r, "{(f,g)}"] \arrow[d, "f"'] \arrow[rrd, bend left, "g"] & X \times Y \arrow[ld, "p"'] \arrow[rd, "q"] & \\
X & & Y
\end{tikzcd}$$

Product of subsets:

is the intersection !!

$$S, T \subseteq \mathcal{U}$$

$$\begin{array}{ccc} W & \subseteq & Z \\ & \rotatebox{90}{$\subseteq$} & \qquad \rotatebox{90}{$\subseteq$} \\ & S & \qquad T \end{array}$$

$$Z \subseteq S, \; Z \subseteq T$$

Co-product: reversed arrows

$$
\begin{tikzcd}
W & & C \arrow[ll] & \\
& X \arrow[ul] \arrow[ur, "i"] & & Y \arrow[ul, "j"'] \arrow[ulll, bend right]
\end{tikzcd}
$$

In sets: $X \sqcup Y$ is the coproduct.

In subsets: union.

$$
\begin{tikzcd}
\supseteq & C & \\
S \arrow[ur, phantom, "\subseteq"] & & T \arrow[ul, phantom, "\supseteq"]
\end{tikzcd}
$$

$t_{13} \circ t_{23} = (1\;3\;2)$

$(1\;3)(2\;3)$

$= (1\;3\;2)$

$t_{13} \circ (1\;2\;3)$

$(1\;3)(1\;2\;3) = (1\;2)$

$= t_{12}$