4800-9

Categories

Sets $\qquad f: S \to T$ |

Subsets $\qquad S \subseteq T$ |

Abelian Gps $\qquad$ linear functions |

Metric Spaces $\qquad$ distance decreasing fcns. |

Vector Spaces/F $\qquad$ F-linear functions |

Groups

Inner Product Spaces

F field (usually $\mathbb{R}$ or $\mathbb{C}$)

- V vector space$/F$
  - $\vec{v}+\vec{w}$
  - $c\vec{v}$

    scalar (i.e. element of F)

- F-linear map:

$$f(\vec{v}+\vec{w}) = f(\vec{v})+f(\vec{w})$$

$$f(c\vec{v}) = c\,f(\vec{v}).$$

Standard Vector Spaces

$$F^n = F \times \cdots \times F$$

f $$= \{(a_1, \dots, a_n) \mid a_i \in F\}$$

$\mathrm{Hom}_F(F^m, F^n)$ is the vector space of $n \times m$ matrices.

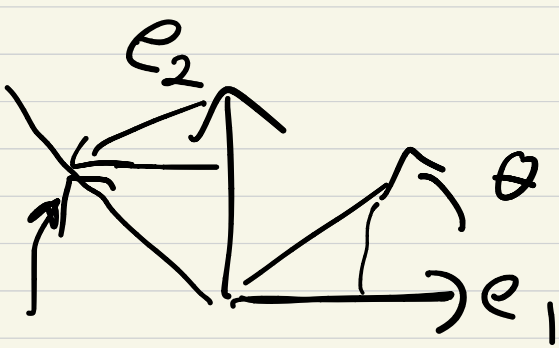
column vectors

$$A = \big(f(e_1) \cdots f(e_m)\big)$$

$$f(x_1, x_2, \dots, x_m)$$

$$= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

E.g. Rotation: by θ



$f : \mathbb{R}^2 \to \mathbb{R}^2$

$$-\sin(\theta)$$
$$\shortparallel$$

$$f(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{pmatrix}$$

$$\cos(\theta) \,\shortparallel$$

$$\hom_F(V, F) = V^\vee$$

(dual)

(vector space F^1)

Rmk: $\hom_F(F^n, F) \simeq F^n$

$F^n \ni e_1, \dots, e_n$ standard vectors

$\vec{v} = v_1 e_1 + \dots + v_n e_n$

$(F^n)^\vee \ni x_1, \dots, x_n$ standard dual vectors

$x_i(\vec{v}) = v_i$

Given $f : F^n \to F$,

then $f = f(e_1)x_1 + \cdots + f(e_n)\cdot x_n$

$$f(\vec{v}) = v_1 f(e_1) + \cdots + v_n f(e_n)$$

$$= x_1(\vec{v}) f(e_1) + \cdots + x_n(\vec{v}) f(e_n)$$

$$f = f(e_1)\, x_1 + \cdots + f(e_n) x_n$$

If $f: V \to W$

then $f^\vee : W^\vee \to V^\vee$

$f^\vee(g: W \to F) = (g \circ f): V \to F$

When $f: F^n \to F^m$

then $f^\vee : (F^m)^\vee \to (F^n)^\vee$

is the transpose matrix.

Familiar Ideas: Given V.

- $W \subseteq V$ is a subspace if W is closed under sums, negatives & scalar multiplication.

- If $\vec{w}_1, \ldots, \vec{w}_m \in V$, then the span of $\vec{w}_1, \ldots, \vec{w}_n$ is:

$$W = \{ c_1\vec{w}_1 + \ldots + c_m\vec{v}_m \}.$$

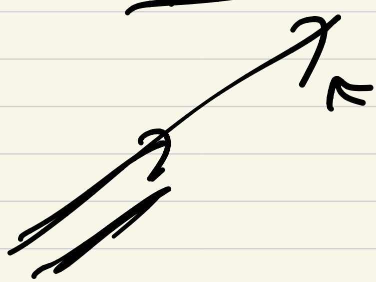
$\vec{w}_1, \dots, \vec{w}_m$ are linearly ind.

if $\quad c_1\vec{w}_1 + \dots + c_m\vec{w}_m = \vec{0}$

$\Leftrightarrow \quad c_1, \dots, c_m = 0$

independent

dependent



If $\vec{w}_1, \ldots, \vec{w}_m$ are linearly independent, then:

$$f : F^m \longrightarrow W = \text{span}$$

$$f(e_1) = \vec{w}_1$$

$$\vdots$$

$$f(e_m) = \vec{w}_m$$

$$f(c_1, \ldots, c_m) = c_1\vec{w}_1 + \cdots + c_m\vec{w}_m$$

is an isomorphism

$f(v)=f(w) \Rightarrow f(v-w)=0$

Rmk: $f: V \to W$ ✓

is injective $\Leftrightarrow$ $f^{-1}(0)=0$.

In general, $f^{-1}(0) \subseteq V$

is a subspace. Called

the kernel of f.

Suppose $f^{-1}(0)=0$. Then $\vec{v}_1=\vec{v}_2$ ✓

$f(\vec{v}_1)=f(\vec{v}_2) \Rightarrow f(\vec{v}_1-\vec{v}_2)=0 \Rightarrow \vec{v}_1-\vec{v}_2=0$

First big idea: dimension.

A set of vectors

$\vec{v}_1, \dots, \vec{v}_n \in V$ is

a basis if

$\text{Span}(\vec{v}_1, \dots, \vec{v}_n) = V$

and

$\vec{v}_1, \dots, \vec{v}_n$ are linearly independent.

Conclusion: A basis of V

determines an isomorphism

$$\left[\ f : F^n \xrightarrow{\sim} V\ \right]$$

$$f(e_1) = \vec{v}_1$$

$$\vdots$$

$$f(e_n) = \vec{v}_n$$

Need to know:

$$\underline{F}^n \not\cong \underline{F}^m \quad \text{if } n \neq m.$$

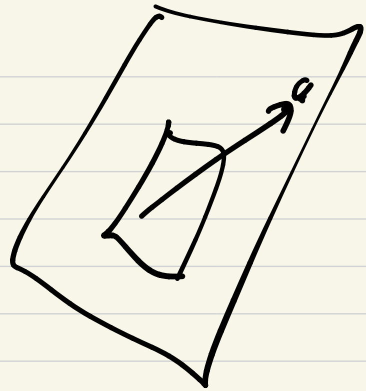
Pf: In F^m, you cannot $> m$ linearly independent vectors.

Row operations:

$$
\begin{bmatrix}
\vec{v}_1 = a_{11} e_1 + \dots + a_{1m} e_m \\
\vdots \\
\vec{v}_{m+1} = \cdot \quad \rule{2em}{0.4pt} \quad - \quad \cdot
\end{bmatrix}
$$

Any subspace

$W \subseteq F^n$



has dimension $\leq n$,

and $= n \Leftrightarrow \underline{W = F^n}$.

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(1) <u>dimension</u>

(2) Given a subspace $W \subset V$,

form the vector space

$$V/W = \{ v + W \}$$

<u>cosets</u>

then

$$\boxed{\dim(W) + \dim(V/W) = \dim(V)}$$

<u>Idea:</u> Start w/ a basis of W

$\vec{w}_1, \ldots, \vec{w}_m$, augment with basis of V/W

$\vec{v}_{m+1} + W, \ldots, \vec{v}_n + W$

$\vec{w}_1, \ldots, \vec{w}_m, \vec{v}_{m+1}, \ldots, \vec{v}_n$ basis for V.

$$f : V \to W$$

$$\ker(f) = f^{-1}(0)$$

$$\left(\operatorname{im}(f) = f(V) = \right)$$

$$\operatorname{coker}(f) = W / f(V)$$

$$\dim(\ker(f)) - \dim(\operatorname{coker}(f))$$

$$= \dim(V) - \dim(W)$$

A **symmetry** $f: F^n \to F^n$

is given by an $n \times n$

matrix

$$A = \left(f(e_1) \; \cdots \; f(e_n) \right)$$

Thm: There is a

$$\det : (n \times n \text{ matrices}) \longrightarrow F$$

such that:

(i) $\det(I_n) = 1$, (ii) $\det(AB) = \det(A)\det(B)$

(iii) det is alternating, i.e.

$$\det(f(e_1) \overset{\text{tr.}}{\curvearrowleft} \cdots f(e_n)) \text{ switches sign.}$$

(iv) det is a tensor.

A tensor is a map

$$T : V \times \cdots \times V \longrightarrow F$$

that is multilinear; i.e.

linear in each vector.

$$T(\vec{v}_1, \dots, \vec{v}_i + \vec{w}_i, \dots, \vec{v}_n)$$

$$= T(\vec{v}_1, \dots, \vec{v}_i, \dots \vec{v}_n) + T(\vec{v}_1, \dots, \vec{w}_i, \dots, \vec{v}_n)$$

<u>Q</u>: <u>How many</u> tensors <u>are there</u>?

$$T(\underline{a_{11} e_1 + a_{21} e_2}, \underline{a_{12} e_1 + a_{21} e_2})$$

$$T(a_{11}e_1 + a_{21}e_2,\ a_{21}e_1 + a_{22}e_2)$$

$$= a_{11}\,T(e_1,\ a_{12}e_1 + a_{22}e_2)$$

$$+ a_{21}\,T(e_2,\ a_{12}e_1 + a_{22}e_2)$$

$$= a_{11}\cdot a_{12}\,\cancel{T(e_1,e_1)}\quad 0$$

$$+ \underline{a_{11}}\,\underline{a_{22}}\,\underline{T(e_1,e_2)}$$

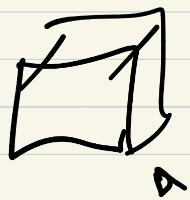
$$+ \underline{a_{21}}\,\underline{a_{12}}\,\underline{T(e_2,e_1)}$$

$$+ a_{21}a_{22}\,\cancel{T(e_2,e_2)}\quad 0$$

$$1 = \underline{T(e_1,e_2)}$$

Think of $\det\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$

$$\boxed{\underline{\underline{\det}} : F^n \times \cdots \times F^n \longrightarrow F}$$



Formula for det.

If $A = (a_{ij})$, then

$$\det(A) = \sum_{\sigma : [n] \to [n]} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$\det(A) = \sum_{\sigma: [n] \to [n]} \operatorname{sgn}(\sigma) \cdot \prod a_{i\,\sigma(i)}$$

2×2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\underset{\sigma}{(id)} = (1)(2)$$

$$\sigma = (1\ 2)$$

$$\det(A) = \underbrace{\operatorname{sgn}(id)}_{=1} \cdot a_{11} \cdot a_{22} \qquad = a_{11}a_{22}$$

$$+ \underbrace{\operatorname{sgn}(12)}_{=-1} \cdot a_{12}\, a_{21} \qquad - a_{12}a_{21}$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\sigma : [3] \to [3]$

$$\det(A) = \overset{+}{\mathrm{sgn}}(\mathrm{id}) \cdot a_{11} a_{22} a_{33}$$

$$+ \overset{-}{\mathrm{sgn}}(12)\, a_{12} a_{21} a_{33}$$

$$+ \overset{-}{\mathrm{sgn}}(13)\, a_{13} a_{22} a_{31}$$

$$+ \overset{-}{\mathrm{sgn}}(23)\, a_{11} a_{23} a_{32}$$

$$+ \overset{+}{\mathrm{sgn}}(123)\, a_{12} a_{23} a_{31}$$

$$+ \overset{+}{\mathrm{sgn}}(132)\, a_{13} a_{32} a_{21}$$

Two deep facts about det

(1) $\det(A^T) = \det(A)$

(2) $\det$ is an <u>alternating</u> <u>tensor</u>.